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ON THE NUMBER OF ISOMORPHISM CLASSES OF CERTAIN TYPES OF ACTIONS AND BINARY SYSTEMS

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Introduction

While in [3] we counted a.o. the number $K(m,n)$ of isomorphism classes of actions Y_X defined on sets $X\times Y$ with $|X|=m$ and $|Y|=n$, and the number of isomorphism classes of binary systems $B(n)$ defined on sets X with $|X|=n$, in [2] we counted a variety of classes of finite posets. Some of the counting techniques developed there have ready applications and variations which are useful in a variety of other contexts. By way of illustration we shall develop some formulas for special classes of actions and binary systems. We begin with some definitions and a listing of information which we shall assume as known.

An action of X on Y is a function $f: X\times Y\to Y$. We shall usually denote actions by Y_X and $f(x,y)=xy$. The set X is considered to be the set of scalars. Another way to view actions is as follows. Let Y be the set of vertices of a polychromatic directed graph and let X be a set of colours. If $xy_1=y_2$, then we envisage this as representating an arrow of colour x as proceeding from y_1 to y_2. In this sense actions are special types of graphs i.e., a polychromatic directed graph is an action provided there is precisely one arrow of each colour $x\in X$ departing from each vertex $y\in Y$. More generally one gets involved with partial actions, where f has domain a subset of $X\times Y$ and various other refinements. As with modules, if A_X and B_X are actions on the same set, then $\phi: A\to B$ is a homomorphism provided $\phi(xa)=x\phi(a)$. A homomorphism ϕ which is also a bijection is an isomorphism.

For actions we define the following operations, which correspond to rather natural constructions in a variety of cases.

Sum: $Y_X+V_U=T_S$, where $T=Y\cup V$ (disjoint union), $X=(X\cup U)$ (disjoint union) and $xv=v$, $uy=y$. If $xy=y$ and $uv=v$ for all y and for all v respectively, then we shall identify x and u.

Ordinal sum: $Y_X\oplus V_U=T_S$, where $T=Y\cup V$ (disjoint union), $S=X\cup V\cup U$

(disjoint union), $xv=v, vy=v, uy=y, vv'=v'$. If $xy=y$ and $uv=v$ for all y and for all v respectively, then we shall identify x and u.

Product: $Y_X V_U = T_S$, where $T=Y\times V$, $S=X\times U$, $(x,u)(y,v)=(xy,uv)$.

Selective product: $Y_X \cdot V_U = T_S$, where $T=Y\times V$, $S=X\cup U$ (disjoint union), $x(y,v)=(xy,v)$, $u(y,v)=(y,uv)$. If $xy=y$ and $uv=v$, x and u are identified.

An action is *sum-primitive* (*S-primitive*), *OS-primitive, P-primitive, SP-primitive* respectively, if it cannot be written as a sum, ordinal sum, product or selective product respectively.

An action Y_X is *faithful* if $xy=x'y$ for all $y\in Y$ implies $x=x'$. Hence, if Y_X and V_U are faithful, then it is easily seen that their sum, ordinal sum, product and selective product are also faithful. An action Y_X is unitary provided for some $x\in X$, $xy=y$ for all $y\in Y$.

For binary systems B, we define the following operations:

$LS(B_1, B_2)=B$, $B=B_1\cup B_2$ (disjoint union) and $b_i\in B_i$ implies $b_1b_2=b_2, b_2b_1=b_1$;
$RS(B_1, B_2)=B$, $B=B_1\cup B_2$ (disjoint union) and $b_i\in B_i$ implies $b_1b_2=b_1$, $b_2b_1=b_2$;
$MS(B_1, B_2)=B$, $B=B_1\cup B_2$ (disjoint union) and $b_i\in B_i$ implies $b_1b_2=b_2$, $b_2b_1=b_2$;
$RMS(B_1, B_2)=B$; $B=B_1\cup B_2$ (disjoint union) and $b_i\in B_i$ implies $b_1b_2=b_1, b_2b_1=b_1$.

We shall call these operations *left sum, right sum, middle sum* and *reverse middle sum* respectively. Again, these various sums correspond to a variety of rather natural constructions. There are of course a host of other possibilities, but we shall only look at these. Again we have a notion of primitivity, with B *LS-primitive* if it is not of the form $B=LS(B_1, B_2)$, with the notions *RS-primitive, MS-primitive* and *RMS-primitive* defined similarly. Finally, a binary system is *primitive* if it is *LS-*, *RS-*, *MS-* and *RMS*-primitive simultaneously.

If B is a binary system such that $B=B_1\cup\ldots\cup B_k$, where $B_i\neq\phi$, $B_iB_i\leq B_i$, $k\geq 2$, and $B_i\cup B_j$ is one of $LS(B_i, B_j)$, $RS(B_i, B_j)$, $MS(B_i, B_j)$ or $RMS(B_i, B_j)$, then B is *decomposable*. B is *indecomposable* if it is not decomposable.

If we take for granted the numbers $K(m, n)$ and $B(n)$, then in this paper we shall essentially determine the number of isomorphism classes of the various types of primitive actions and the various types of primitive binary systems. We shall also determine the number of isomorphism classes of indecomposable binary systems. We recall that $T(k, n)$, the number of actions Y_X such that $X\subset Y^Y$, $|X|=k$, $|Y|=n$ has also been determined in [3].

On the number of faithful S-primitive actions.

If $T(n)=\sum_{k=1}^{n^n} T(k,n)$, then $T(n)$ is determined and represents the total number of actions Y_X with $X\subset Y^Y$, $|Y|=n$ and $|X|\geq 1$. Since $X\subset Y^Y$, the action Y_X is faithful, and conversely if Y_X is a faithful action, then $x\to f_x$, where $f_x y=xy$, is an injection of X into Y^Y, so that in fact the condition that Y_X is faithful is essentially equivalent to our taking X to be a subset of Y^Y. Therefore $T(n)$ represents the total number of faithful actions Y_X with $|Y|=n$.

Let $\{\alpha_i|i\in\omega\}$ denote the collection of finite S-primitive faithful actions. Let $P=\{\sum n_i\alpha_i|n_i\in\{0,1,2,\dots\}\}$ denote the collection of finite linear combinations of finite S-primitive faithful actions. Let $f^*(\alpha_i)=|Y|$ if $\alpha_i=Y_X$. Let $f(\sum n_i\alpha_i)=\sum n_i f^*(\alpha_i)$. Let $T^*(n)=|\{\alpha_i|f^*(\alpha_i)=n\}|$, $T'(n)=|\{\sum n_i\alpha_i|f(\sum n_i\alpha_i)=n\}|$. Obviously $T'(n)$ is identical to the number $T(n)$ defined above. We're interested in determining the number $T^*(n)$ of S-primitive faithful actions Y_X such that $|Y|=n$.

If $T_0(k,n)$ denotes the number of faithful unitary actions Y_X with $|X|=k$, $|Y|=n$, then by adjoining identity maps we find that:

(1) $T(k,n)-T_0(k,n)=T_0(k+1,n)$.

Hence, if $T_0(n)=\sum_k T_0(k,n)$, then:

(2) $T(n)=\sum_k T_0(k,n)+\sum_k T_0(k+1,n)=2T_0(n)-T_0(1,n)=2T_0(n)-1$.

Also, we have:

(3) $T_0(k,n)=\sum_{i=0}(-1)^i T(k+i,n)$, so that the numbers $T_0(n)$ and $T_0(k,n)$ are determined.

If $T_0{}^*(n)$ and $T_0{}^*(k,n)$ denote the number of S-primitive faithful unitary actions Y_X with $|Y|=n$ (resp. $|X|=k$, $|Y|=n$), then since adding or removing an identity map will not affect the S-primitivc of an action, we have:

(4) $T_0{}^*(k,n)+T_0{}^*(k+1,n)=T^*(k,n)$, the number of S-primitive faithful actions Y_X with $|X|=k$, $|Y|=n$.

Hence, it follows that:

(5) $T^*(n)=\sum_k T^*(k,n)=2T_0{}^*(n)-T_0{}^*(1,n)=2T_0{}^*(n)-1$, as in formula (2) for $T(n)$ and $T_0(n)$.

Summing is a commutative operation, i.e., $Y_X+V_U=V_U+Y_X$ in the sense of identical actions and hence certainly if we use = to denote isomorphic actions.

In the following we use constructions also discussed in [2], so for other examples we refer the reader to that paper.

Let $W=\{1, 2, \dots\}$ be the set of positive integers. Define products $W^k \times W^k \to W$ as follows:

(6) $(e_1, \cdots, e_k) \cdot (n_1, \cdots, n_k) = e_1 n_1 + \cdots + e_k n_k$ and $(e_1, \cdots, e_k) \times (n_1, \cdots, n_k) = n_1^{(e_1)} \cdots n_k^{(e_k)}$, where $n^{(e)} = \binom{n}{1}\binom{e-1}{0} + \cdots + \binom{n}{e}\binom{e-1}{e-1} = \binom{n+e-1}{e}$.

Thus $n^{(e)}$ is also equal to the number of ordered partitions of $n-1$ into $e+1$ non-negative integers.

If $f: W \to W$ is any function whatsoever, we define

(7) $(e_1, \cdots, e_k) \times_f (n_1, \cdots, n_k) = (e_1, \cdots, e_k) \times (f(n_1), \cdots, f(n_k))$.

Thus, if $n_1 > n_2 > \cdots > n_k$, then $(e_1, \cdots, e_k) \times_{T_0^*} (n_1, \cdots, n_k)$ is the number of faithful unitary actions having $e_1 + \cdots + e_k$ S-primitive faithful unitary components, of which precisely e_i components have n_i elements.

That this is indeed the case can be seen as follows. Suppose that we consider all faithful unitary actions having e S-primitive faithful unitary components each containing n elements. Then, if we assume that there are k-non-isomorphic types present, then these can be distributed in $p_k(e) = \binom{e-1}{k-1}$ ways, where $p_k(e)$ is the number of ordered partitions of e into k positive integers, each such partition corresponding to a distinct distribution of these k types. Now, if $T_0^*(n) = m$, then these k types can be selected in $\binom{m}{k}$ ways, disregarding order. Hence the total number of actions which can be constructed from k types is $\binom{m}{k}\binom{e-1}{k-1}$. Summing over k, we find that we end up with $m^{(e)}$ possibilities. The more general case follows at once, since actions having different cardinalities cannot be isomorphic, i.e., the product rule holds in that situation when considering the decomposition of the faithful unitary action into its e_i components each containing n_i elements for $i = 1, \cdots, k$.

We shall call a vector $\vec{n}$ in W^k *cascading* if $\vec{n} = (n_1, \cdots, n_k)$ with $n_1 > n_2 > \cdots > n_k$. A *bipartition* of m is a pair $(\vec{e}, \vec{n}) \in W^k \times W^k$ for some k such that $\vec{n}$ is cascading and such that $\vec{e} \cdot \vec{n} = m$.

Hence, if we consider the sum over all bipartitions of m $\sum_{\vec{e} \cdot \vec{n} = m} \vec{e} x_{T_0^*} \vec{n}$, then this number is $T_0(m)$.

Now let $g_1(X_1)=X_1$, $\vec{e}\otimes\vec{N}=g_{N_1}(x_1, \cdots, x_{N_1})^{(e_1)}\cdots g_{N_k}(x_1, \cdots, x_{N_k})^{(e_k)}$ and $g_n(X_1, \cdots, X_n)=X_n-\sum^*_{\vec{e}\cdot\vec{N}=n}\vec{e}\times\vec{N}$ where $\sum^*$ indicates that we delete the bipartition $(1)\cdot(n)=n$, and where $p(x)^{(e_1)}=\binom{p(x)+e-1}{e}=1/e!(p(x)+e-1)(p(x)+e-2)\dots(p(x)+1)p(x)$ for $p(x)=p(x_1, \cdots, x_s)$ any polynomial in any finite number of indeterminates. Then it follows that

(8) $$T_0^*(n)=g_n(T_0(1), \cdots, T_0(n)),$$

as can be seen by simple substitution of the formulas for $T_0(i)$, $1\leq i\leq n$ into $g_n(x_1, \cdots, x_n)$. Indeed, if f and y are functions which are related by an equation:

(9) $$f(m)=\sum_{\vec{e}\cdot\vec{n}=m}\vec{e}x_h\vec{n},$$

then it is also true that:

(10) $$h(n)=g_n(f(1), \cdots, f(n)).$$

The polynomials $g_n(x_1, \cdots, x_n)$ can be computed recursively and in [2] the first several ones are given. As we shall see below, this type of process has many variants and applications, in this paper we give only a few examples.

In order to determine $T_0^*(k, n)$ we proceed in a similar fashion. If w is the ordered set $\{1, 2, 3, \cdots\}$ then we turn w^2 into an ordered set by letting $a=(a_1, a_2)<b=(b_1, b_2)$ if $a_1+a_2<b_1+b_2$ or if $a_1+a_2=b_1+b_2$ and $a_1<b_1$. Thus, the number of elements preceding any given element is finite. If $\vec{a}=(a_1, \cdots, a_t)$, $a_i\in w^2$, say $\vec{a}$ is cascading if $a_1>a_2>\cdots>a_t\geq(1, 1)$. For $\vec{e}\in w^t$, $\vec{a}\in(w^2)^t$, let $\vec{e}\cdot\vec{a}=((\sum e_ia_{i1})+1-t^*, \sum e_ia_{i2})$, $a_i=(a_{i1}, a_{i2})$, $t^*=e_1+\cdots+e_t$.

Then, by an argument very similar to that used above we establish that:

(11) $T_0(k, n)=\sum_{\vec{e}\cdot\vec{2}=(k,n)}\vec{e}\times_{T_0^*}\vec{a}$, where

(12) $$\begin{aligned}\vec{e}\times_{T_0^*}\vec{a}&=(e_1, \cdots, e_t)\times(T_0^*(a_{11}, a_{12}), \cdots, T_0^*(a_{t1}, a_{t2}))\\&=T_0^*(a_{11}, a_{12})^{(e_1)}\cdots T_0^*(a_{t1}, a_{t2})^{(e_2)}\end{aligned}$$

As before in the sum in (11), $\vec{a}$ runs over all cascading vectors. In particular, $(1)\cdot(k, n)=(k, n)$ and $(1)\times_{T_0^*}(k, n)=T_0^*(k, n)^{(1)}=T_0^*(k, n)$.

Again, if we define the polynomials $g_{(k,n)}(x_{11}, \cdots, x_{kn})$ inductively by the scheme

(13) $g_{(1,1)}(x_{11})=x_{11}$, $g_{(k,n)}(x_{11}, \cdots, x_{kn})=x_{kn}-\sum^*_{\vec{e}\cdot\vec{a}=(k,n)}\vec{e}\otimes\vec{a}$,

where $\sum^*$ indicates that we delete the bipartition $(1)\cdot(k, n)=(k, n)$ and where

(14) $\vec{e}\otimes\vec{a}=g_{(a_{11},a_{12})}(x_{11},\cdots,x_{a_{11}a_{12}})^{(e_1)}\cdots g_{(a_{t1},a_{2t})}(x_{11},\cdots,x_{at,at_2})^{(e_t)}$.

It follows that we can invert formula (11) to obtain

(15) $T_0^*(k,n)=g_{(k,n)}(T_0(1,1),\cdots,T_0(k,n))$

From (14) and (15) we have also determined $T^*(k,n)$, while (5) and (9) give us the number $T^*(n)$.

On the number of faithful OS-primitive actions

The next item on the agenda is to deal with ordinal sums. Here, since for OS-primitive actions Y_X and V_U, $Y_X\oplus V_U=V_U\oplus Y_X$ if and only if $Y_X=V_U$, we have the extreme non-commutative situation as in ordinal sums of posets. The solutions follow the pattern established above using different functions.

Let $F_0^*(k,n)$ denote the number of OS-primitive faithful unitary actions Y_X, with $|X|=k$, $|Y|=n$. Let $F_0^*(n)=\sum F_0^*(k,n)$.

From the non-commutativity of ordinal sums in the sense described above, we establish in a straightforward manner that

(16) $T_0(n)=\sum_{\vec{e}\cdot\vec{N}=n}\vec{e}\wedge_{F_0^*}\vec{N}$, where

(17) $(e_1,\cdots,e_t)\wedge(N_1,\cdots,N_t)=N_1^{e_1}\cdots N_t^{e_t}\binom{e_1+\cdots+e_t}{e_1,\cdots,e_t}$, and

(18) $(e_1,\cdots,e_t)\wedge_f(N_1,\cdots,N_t)=(e_1,\cdots,e_t)\wedge(f(N_1),\cdots,f(N_t))$.

Here as usual in (16) we sum only over bipartitions.

To invert formula (16) we define polynomials $G_n(x_1,\cdots,x_n)$ inductively by $G_1(x_1)=x_1$ and

(19) $G_n(x_1,\cdots,x_n)=x_n-\sum_{\vec{e}\cdot\vec{N}=n}\vec{e}\circledcirc\vec{N}$,

(20) $(e_1,\cdots,e_t)\circledcirc(N_1,\cdots,N_t)=G_{N_1}(x_1,\cdots,x_{N_1})^{e_1}\cdots G_{N_t}(x_1,\cdots,x_{N_t})^{e_t}\binom{e_1+\cdots+e_t}{e_1,\cdots,e_t}$.

Again in (19) $\sum^*$ indicates that the bipartition $(1)\cdot(n)=n$ is deleted. It follows that the polynomials $G_n(x_1,\cdots,x_n)$ can be computed recursively and that

(21) $F_0^*(n)=G_n(T_0(1),\cdots,T_0(n))$.

In order to compute $F_0^*(k,n)$ we take a product on w^2 defined as follows:

(22) $\vec{e}\cdot\vec{a}=(e_1,\cdots,e_t)\cdot((a_{11},a_{12}),\cdots,(a_{t1},a_{t2}))$
$=(\sum e_i(a_{i1}+a_{i2})+1-a_{t2}-t^*, \sum e_i a_{i2})$.

A typical argument shows that:

(23) $T_0(k,n)=\sum_{\vec{e}\cdot\vec{a}=n}\vec{e}\wedge_{F_0^*}\vec{a}$, where

(24) $\vec{e}\wedge_{F_0^*}\vec{a}=F_0^*(a_{11}, a_{12})^{e_1}\cdots F_0^*(a_{t1}, a_{t2})^{e_t}\binom{e_1+\cdots+e_t}{e_1, \cdots, e_t}.$

Thus if we take $G_{(1,1)}(x_{11})=x_{11}$ and

(25) $G_{(k,n)}(x_{11}, \cdots, x_{kn})=x_{kn}-\sum^*_{\vec{e}\cdot\vec{a}=n} \vec{e}\circledwedge\vec{a},$

(26) $\vec{e}\circledwedge\vec{a}=G_{(a_{11},a_{12})}(x_{11}\cdots,)^{e_1}\cdots G_{(a_{t1},a_{t2})}(x_{11}, \cdots)^{e_t}\binom{e_1+\cdots+e_t}{e_1, \cdots, e_t}.$

With $\sum^*_{\vec{e}\cdot\vec{a}}$ running over all bipartitions except $(1)\cdot(k, n)=(k, n)$ we have the inversion we need. Hence

(27) $F_0^*(k, n)=G_{(k,n)}(T_0(1, 1), \cdots, T_0(k, n)).$

If we let $F^*(n)$ denote the number of faithful OS-primitive actions Y_X such that $|Y|=n$, then as before we observe that adding an identity map does not change OS-primitivity, and hence as above $F^*(n)=2F_0^*(n)-1$. Also, if $F^*(k, n)$ denotes the number of faithful OS-primitive actions Y_X such that $|Y|=n$, $|X|=k$, then as in (4) we have a relation $F_0^*(k, n)+F_0^*(k+1, n)=F^*(k, n)$. Thus the quantities $F^*(n)$ and $F^*(k, n)$ have also been determined.

On the number of faithful P-primitive actions

For products we are back in the commutative situation. Also, since Y_XV_U is unitary and faithful if and only if Y_X and V_U are both unitary and faithful, then we may compute certain coefficients without first passing to unitary actions. Thus let $P_0^*(k, n)$ denote the number of faithful unitary P-primitive actions Y_X such that $|X|=k$, $|Y|=n$. Let $P_0^*(n)$ denote the number of faithful unitary P-primitive actions Y_X such that $|Y|=n$. Let $P^*(k, n)$ denote the number of faithful P-primitive actions Y_X such that $|Y|=n$ and $|X|=k$. Finally let $P^*(n)$ denote the number of faithful P-primitive actions Y_X such that $|Y|=n$. Obviously, $P_0^*(n)=\sum_k P_0^*(k, n)$ and $P^*(n)=\sum_k P^*(k, n)$.

Now, we use a scalar product

(28) $(e_1, \cdots, e_t)\cdot(N_1, \cdots, N_t)=N_1^{e_1}\cdots N_t^{e_t}$, and the vector product

(29) $(e_1, \cdots, e_t)\times(N_1, \cdots, N_t)=N_1^{(e_1)}\cdots N_t^{(e_t)}.$

A bipartition $\vec{e}\cdot\vec{N}=n$ requires $\vec{N}$ to be a cascading vector. In a straightforward manner we find

(30) $T_0(n)=\sum_{\vec{e}\cdot\vec{N}=n} \vec{e}\times_p\vec{N},\quad T(n)=\sum_{\vec{e}\cdot\vec{N}=n} \vec{e}\times_p\vec{N}.$

Here the summation runs over bipartitions $e\cdot\vec{N}=n$ as usual.

If we let $h_1(x_1)=x_1$ and

(31) $h_n(x_1, \cdots, x_n)=x_n-\sum^*_{\vec{e}\cdot\vec{N}=n} \vec{e}\otimes\vec{N}$, with $\sum^*$ indicating that $(1)\cdot(n)=n$ is excluded, then it follows that

(32) $P_0^*(n)=h_n(T_0(1), \cdots, T_0(n))$, $P^*(n)=h_n(T(1), \cdots, T(n))$.

For $P_0^*(k, n)$ and $P^*(k, n)$ we work with bipartitions of elements in w^2 and we use the scalar product:

(33) $$\begin{aligned}\vec{e}\cdot\vec{a}&=(e_1, \cdots, e_t)\cdot((a_{11}, a_{12}), \cdots, (a_{t1}, a_{t2}))\\ &=((e_1, \cdots, e_t)\cdot(a_{11}a_{21}, \cdots, a_{t1}), (e_1, \cdots, e_t)\cdot(a_{12}, a_{22}, \cdots, a_{t2}))\\ &=(\prod_{i=1}^{t} a_{i1}^{e_i}, \prod_{i=1}^{t} a_{i2}^{e_i}).\end{aligned}$$

Another argument of the standard type and we find

(34) $T_0(k, n)=\sum_{\vec{e}\cdot\vec{a}=(k, n)} \vec{e}\times_{p_0*}\vec{a}$, $T(k, n)=\sum_{\vec{e}\cdot\vec{a}=(k, n)} \vec{e}\times_{p*} a$,

Thus to invert we define $h_{(1,1)}(x_{11})=x_{11}$ and

(35) $h_{(k,n)}(x_{11}, \cdots, x_{kn})=x_{kn}-\sum^*_{\vec{e}\cdot\vec{a}=(k, n)} \vec{e}\otimes\vec{a}$.

with $(1)\cdot(k, n)=(k, n)$ deleted in $\sum^*$ as usual. It follows that

(36) $$\begin{aligned}P_0^*(k, n)&=h_{(k, n)}(T_0(1, 1), \cdots, T_0(k, n))\\ P^*(k, n)&=h_{(k, n)}(T(1, 1), \cdots, T(k, n)).\end{aligned}$$

On the number of faithful *SP*-primitive actions

For selective products we are again dealing with a commutative situation. Because of the definition we work with unitary actions and numbers $S_0^*(k, n)$ and $S_0^*(n)$, where the first number is the number of faithful unitary *SP*-primitive actions Y_X with $|Y|=n$ and $|X|=k$, and where the second number is the number of faithful unitary *SP*-primitive actions Y_X with $|Y|=n$.

If we define $\vec{e}\cdot\vec{N}$ as in (28) and $\vec{e}\times\vec{N}$ as in (29), then

(37) $$\begin{aligned}T_0(n)&=\sum_{\vec{e}\cdot\vec{N}=n} \vec{e}\times_{S_0*}\vec{N}\\ S_0^*(n)&=h_n(T_0(1), \cdots, T_0(n)).\end{aligned}$$

Hence, $P_0^*(n)=S_0^*(n)$, which from the definitions seems reasonable enough.

For $S_0^*(k, n)$ we define the scalar product

(38) $\vec{e}\cdot\vec{a}=(e_1,\cdots,e_t)\cdot((a_{11},a_{12}),\cdots,(a_{t1},a_{t2}))$
$=((\sum e_i a_{i1})+1-t^*,\ \prod_{i=1}^{t} a_{12}^{e_i}),$ and

(39) $T(k,n)=\sum_{\vec{e}\cdot\vec{a}=(k,n)} \vec{e}\times_{S_0^*}\vec{a}.$

For formula (38) compare the definition with formula (33) and the definition preceding formula (11).

Now, letting $f_{(1,1)}(x_{11})=x_{11}$ and

(40) $f_{(k,n)}(x_{11},\cdots,x_{kn})=x_{kn}-\sum^*_{\vec{e}\cdot\vec{a}=(k,n)} \vec{e}\otimes\vec{a},$

then

(41) $S_0^*(k,n)=f_{(k,n)}(T(1,1),\cdots,T(k,n)).$

From the definition of SP-primitivity it follows that $S^*(k,n)=S_0^*(k,n)+S_0^*(k+1,n)$ determines the number of faithful SP-primitive actions Y_X such that $|Y|=n$, $|X|=k$. Hence, $S^*(n)=\sum_k S^*(k,n)=2S_0^*(n)-S_0^*(1,n)$ is the number of faithful SP-primitive actions Y_X such that $|Y|=n$. Now $S_0^*(1,n)=1$ if n is a prime or 1, $S_0^*(1,n)=0$ otherwise. Thus n is a composite number if and only if $S^*(n)$ is even. A rather inefficient test, to be sure.

Dropping the faithfulness requirements

If Y_X is any action whatsoever, then X determines a set $X^*\subset Y^Y$ by mapping $x\to f_x: Y\to Y$, where $f_x y=xy$. Now, if we have a sum $Y_X+V_U=T_S$, then for S^* we have a decomposition $S^*=X^*\cup U^*$ with the proper extension of the definition of the mappings in X^* and U^*. In other words, if $Y_X+Y_U=T_S$, then $Y_{X^*}+V_{U^*}=T_{S^*}$, and hence Y_X is S-primitive if and only if Y_{X^*} is S-primitive.

If $p_k(m)$ denotes the number of ordered partitions of m into k parts and if $|X|=m$, $|X^*|=k$, then there are $P_k(m)$ ways that X may generate X^*.

Here $p_k(m)=\binom{m-1}{k-1}$.

Thus if we let $K^*(m,n)=\sum_{k=1}^{m} p_k(m)T^*(k,n)$, then $K^*(m,n)$ denotes the number of S-primitive actions Y_X such that $|Y|=n$ and $|X|=m$.

If we want to preserve the requirement that the action Y_X be unitary, then since Y_{X^*} is also unitary, we merely take the sum $K_0^*(m,n)=\sum_{k=1}^{m} p_k(m)T_0^*(k,n)$

to obtain the number of unitary S-primitive actions Y_X such that $|Y|=n$ and $|X|=m$.

If we consider an ordinal sum $Y_X+V_U=T_S$, then $S^*=X^*\cup V^*\cup U^*$ in the appropriate way and where $f_v=f_{v'}$ implies $f_v y=v=f_{v'}y=v'$ and $v=v'$. Thus there is a natural correspondence between $Y_{X^*}+V_{U^*}$ and T_{S^*}, so that again Y_X is OS-primitive if and only if Y_{X^*} is OS-primitive. Hence, we find that $L^*(m, n) = \sum_{k=1}^{m} P_k(m)F^*(k, n)$ denotes the number of OS-primitive actions Y_X such that $|Y|=n$ and $|X|=m$. If we want to restrict ourselves to unitary actions only, then since we have already seen that Y_X is unitary if and only if Y_{X^*} is unitary, it suffices to take $L_0^*(m, n)=\sum_{k=1}^{m} p_k(m)F_0^*(k, n)$ to obtain the number of unitary OS-primitive actions Y_X such that $|Y|=n$ and $|X|=m$.

For products $Y_X V_U=T_S$, we have $S^*=X^*\times U^*$ in a natural way, i.e., $T_{S^*}=Y_{X^*}V_{U^*}$, whence Y_X is P-primitive if and only if Y_{X^*} is P-primitive and we have corresponding formulas $M^*(m, n)=\sum_{k=1}^{m} p_k(m)P^*(k, n)$ and $M_0^*(m, n)=\sum_{k=1}^{m} P_k(m)P_0^*(k, n)$ to denote the number of P-primitive actions Y_X such that $|Y|=m$ and $|X|=n$ and the number of unitary actions Y_X of the same type respectively.

In the case of selective products $Y_X\cdot V_U=T_S$ we again obtain an obvious isomorphism between T_{S^*} and $Y_{X^*}\cdot V_{U^*}$, so that by the same arguments as before $N^*(m, n)=\sum_{k=1}^{m} p_k(m)S^*(k, n)$ and $N_0^*(m, n)=\sum_{k=1}^{m} p_k(m)S_0^*(k, n)$ denote the number of SP-primitive actions Y_X such that $|Y|=m$ and $|X|=n$ and the number of unitary actions Y_X of the same type which are also unitary.

Fixed points

If O_1 denotes the action $1\cdot 0=0$, then if Y_X is an action with an element 0 such that $x0=0$ for all $x\in X$ and such that $y\neq 0$ implies $xy\neq 0$, then we may write $Y_X=Y_X^*+0_1$, where $Y^*=Y-\{0\}$.

Let $T_{00}(k, n)$ denote the number of faithful unitary actions Y_X without an element of the type 0, $|Y|=n$, $|X|=k$. If $T_{0i}(k, n)$ denotes the number of faithful unitary actions with precisely i elements of the type 0, $|Y|=n$, $|X|=k$, then it follows that $T_{00}(k, n)=T_{0i}(k, n+i)$. Hence it follows that

(42) $T_0(k, n)=T_{00}(k, n)+T_{00}(k, n-1)+\cdots$

From this we compute $T_{00}(k, n) = T_0(k, n) - T_0(k, n-1)$ immediately.

Binary systems

Having completed our program for actions, we shall now concern ourselves with binary systems and the sum operations defined for these. We begin with some observations.

1. If $B = MS(B_1, B_2)$, then $B = RMS(B_2, B_1)$.
2. If $B = LS(B_1, B_2) = RS(B_1^*, B_2^*)$, let $a \in B_1 \cap B_1^*$, $b \in B_2 \cap B_2^*$. Then $ab = b = a$, a contradiction since $B_1 \cap B_2 = \phi$. Hence $B_2 \cap B_2^* = \phi$ without loss of generality, i.e., $B_2^* \subset B_1$, $B_2 \subset B_1^*$. Now let $a \in B_2^*$, $b \in B_2$, then $ab = b = a$, i.e., $B_2^* = B_2$ and $B_2 \cap B_1 \neq \phi$, a contradiction. Hence if $B = LS(B_1, B_2)$ then B is RS-primitive.
3. If $B = RS(B_1, B_2)$ then B is LS-primitive.
4. If $B = LS(B_1, B_2) = MS(B_1^*, B_2^*)$, then let $a \in B_1 \cap B_1^*$, $b \in B_2 \cap B_2^*$. We have $ba = b = a$, a contradiction since $B_1 \cap B_2 = \phi$. Hence, $B_2 \cap B_2^* = \phi$ without loss of generality, i.e., $B_2^* \subset B_1$, $B_2 \subset B_1^*$. If $a \in B_2^*$, $b \in B_2$, then $ba = b = a$, i.e., $B_2^* = B_2$, and $B_2 \cap B_1 = \phi$, a contradiction. Hence if $B = LS(B_1, B_2)$, then B is MS-primitive.
5. If $B = MS(B_1, B_2)$ then B is LS-primitive.
6. If $B = RS(B_1, B_2)$ then B is MS-primitive and if $B = MS(B_1, B_2)$, then B is RS-primitive.
7. The operations are associative. Thus, given B_1, B_2, B_3 we claim equality in the following situations.
   (a) $LS(B_1, LS(B_2, B_3)) = LS(LS(B_1, B_2), B_3)$,
   (b) $RS(B_1, RS(B_2, B_3)) = RS(RS(B_1, B_2), B_3)$,
   (c) $MS(B_1, MS(B_2, B_3)) = MS(MS(B_1, B_2), B_3)$.

   Obviously we have equality of the underlying sets. Now, select $b_i \in B_i$, $i = 1, 2, 3$ and compute six multiplication matrices $(b_i b_j)$, $1 \leq i, j \leq 3$ corresponding to the six situations, comparing them pairwise. Thus e.g., the matrix:

$$
\begin{bmatrix} b_1^2 & b_2 & b_3 \\ b_2 & b_2^2 & b_3 \\ b_3 & b_3 & b_3^2 \end{bmatrix} \tag{43}
$$

corresponds to both sides of (c). Thus it follows that the binary systems on both sides of (c) are equal.

8. From the associativity of the operations it follows that we have essentially unique decompositions $B=LS(B_1, \cdots. B_k)$ with B_i LS-primitive in the case that B is a finite set, and similarly for the other operations. Thus, if $B=LS(B_1, \cdots, B_k)=LS(C_1, \cdots, C_l)$ are two decompositions of B into LS-primitive subsystems, then $A_{ij}=B_i \cap C_j$ is either empty or a subsystem. Furthermore, since $B_1=LS(A_{11}, A_{12}, \cdots, A_{1l})$, where we delete empty sets wherever necessary, it follows that since B_1 is LS-primitive, $B_1=A_{i(1)}$ and $B_1 \subset C_{i(1)}$ for some $i(1)$. Similarly, $C_{i(1)} \subset B_{j(i(1))}$ and by the disjointness of the B_i, $B_1=C_{i(1)}$, so that without loss of generality we may also take $B_1=C_1$, $B_2=C_2$, etcetera. It follows that $k=l$ as well.
9. If we consider a finite B, let $LS \cdot B$ denote the elements $\{B_1, \cdots, B_k\}$ occurring in a decomposition $B=LS(B_1, \cdots, B_k)$, repeated if necessary and let $RS \cdot B$, $MS \cdot B$ be defined in a similar fashion. If for a set $\{B_1, \cdots, B_k\}$ we let $LS \cdot \{B_1, \cdots, B_k\} = \{LS \cdot B_1, LS \cdot B_2, \cdots, LS \cdot B_k\}$ (i.e., the union counting repetitions separately) with $RS \cdot \{B_1, \cdots, B_k\}$ and $MS \cdot \{B_1, \cdots, B_k\}$ similarly defined, then starting with finite B we obtain a sequence

   (44) $B \rightarrow LS \cdot B \rightarrow RS \cdot (LS \cdot B) \rightarrow MS \cdot (RS \cdot (LS \cdot B)) \rightarrow \ldots$

   which must eventually terminate in a set $\{B_1, \cdots, B_k\}$ of elements B_i which are primitive, i.e., LS-primitive, RS-primitive and MS-primitive (and thus also RMS-primitive as in the introduction, by use of comment 1).
10. From the relations between different kinds of primitivity as described in comments 2, 3, 4, 5 and 6, and the idempotence of the mappings, $LS \cdot (LS \cdot B) = LS \cdot B$, etcetera, it follows that the primitive parts of B are also uniquely determined, since in fact the sequence (44) starts for a unique one of the three mappings, i.e., the order of the mappings is essentially immaterial.
11. $LS(B_1, B_2)=LS(B_2, B_1)$ and $RS(B_1, B_2)=RS(B_2, B_1)$ and $MS(B_1, B_2)=MS(B_2, B_1)$ if and only if $B_1=B_2$ for MS-primitive binary systems B_1 and B_2.

   Having noted these facts we are in a position to commence the counting process.

Various numbers

If $B_L(n)$ denotes the number of LS-primitive binary systems B with $|B|=n$ then from comment 11 it follows that

(45) $B(n)=\sum_{\vec{e}\cdot\vec{N}=n} \vec{e}\times_{B_L}\vec{N}$, and $B_L(n)=g_n(B(1),\cdots,B(N))$,

where the polynomials $g_n(x_1,\cdots,x_n)$ are those defined as in formula (8).

If $B_R(n)$ denotes the number of RS-primitive binary systems B with $|B|=n$, then from comment 11 it follows that $B_R(n)=B_L(n)$.

If $B_M(n)$ denotes the number of MS-primitive binary systems B with $|B|=n$, the anticommutative situation applies and

(46) $B(n)=\sum_{\vec{e}\cdot\vec{N}=n} \vec{e}\wedge_{B_L}\vec{N}$ and $B_M(n)=G_n(B(1),\cdots,B(n))$

where the polynomials $G_n(x_1,\cdots,x_n)$ are those defined in formulas (19) and (20).

These being the basic quantities required, we shall next consider various other cases.

Suppose $B_{L,R}(n)$ denotes the number of binary systems B with $|B|=n$ which are both LS-primitive and RS-primitive. Then, from comments 2 and 3, it follows readily that

(47) $B(n)=B_L(n)+B_R(n)-B_{L,R}(n)$, or $B_{L,R}(n)=2B_L(n)-B(n)$.

Similarly, it follows that

(48) $B(n)=B_L(n)+B_M(n)-B_{L,M}(n)$, or $B_{L,M}(n)=B_L(n)+B_M(n)-B(n)$.

where $B_{L,M}(n)$ is the number of binary systems B with $|B|=n$ which are both LS-primitive and MS-primitive.

Finally, if $B_p(n)$ is the number of primitive binary systems B with $|B|=n$, then we have equations

(49) $B(n)=2B_L(n)+B_M(n)-2B_{L,M}(n)-B_{L,R}(n)+B_p(n)$, and

(50) $B_p(n)=2B_L(n)+B_M(n)-2B(n)$.

If we set $B_1\sim B_2$ if B_1 and B_2 have the same primitive parts, and if $C(n)=\sum_{\vec{e}\cdot\vec{N}=n} \vec{e}\times_{B_p}\vec{N}$, then $C(n)$ is the number of equivalence classes $[B_1]$, where $|B_1|=n$ and $[B_1]=\{B_2|B_1\sim B_2\}$. Also, $B_p(n)=g_n(C(1),\cdots,C(n))$.

Another type of decomposition

Suppose B is a binary system such that $B=B_1\cup\cdots\cup B_k$, where $B_iB_i\subset B_i$, $B_i\neq\phi$, and where $B_i\cup B_j$ is one of $LS(B_i,B_j)$, $RS(B_i,B_j)$, $MS(B_i,B_j)$ or $RMS(B_i,B_j)$. Then $\{B_1,\cdots,B_k\}$ is a decomposition of B. Also, B is indecomposable if the only decomposition of B is $\{B\}$. Notice that if B is indecomposable, then it is primitive since $B=LS(B_1, B_2)$ implies $\{B_1, B_2\}$ is a decomposition. Suppose $\{B_1, \cdots, B_k\}$ and $\{C_1, \cdots, C_l\}$ are decompositions of B. Let $A_{ij}=B_i \cap C_j$, then if $A_{ij} \neq \phi$, we certainly have $A_{ij}A_{ij} \subset A_{ij}$. Suppose we consider A_{ij} and A_{rs}, with $(i,j) \neq (r,s)$. Then, $i \neq r$ without loss of generality, and $B_i \cup B_r = LS(B_i, B_r)$ without loss of generality. It follows that then also $LS(A_{ij}, A_{rs})$, so that if $A=\{A_{ij} \mid A_{ij} \neq \phi\}$, then $A=\{B_1, \cdots, B_k\} \cap \{C_1, \cdots, C_l\}$ is also a decomposition of B. In particular, if B is finite, then the intersection of all decompositions of B yields a unique finest decomposition $\{B_1, \cdots, B_k\}$ whose elements are themselves indecomposable. Our next object is to say something about the number $B_I(n)$ of indecomposable binary systems B such that $|B|=n$.

We shall begin by counting those binary systems B with the property that $|B|=k$ and $\{B_1, \cdots, B_k\}$ is a decomposition with $B_i=\{b_i\}$ the singleton system $b_ib_i=b_i$. These systems are quite obviously characterized by the rule $b_ib_j \in \{b_i, b_j\}$. In terms of a colouration associated with B, we consider the set $(k \times k)_0 = \{(i,j) \mid i \neq j\}$, and we colour the elements (i,j) by 0 or 1 according to the rules $X(i,j)=1$ if $b_ib_j=b_j$ and $X(i,j)=0$ if $b_ib_j=b_i$. Conversely, any colouration immediately determines a binary system of the type. We operate with S_k on $(k \times k)_0$ by $\bar{\phi}(i,j)=(\phi(i), \phi(j))$ for $\phi \in S_k$. Then it follows that if ϕ is of type $1^{\mu_1} \cdots s^{\mu_s}$, then $\bar{\phi}$ is of type $1^{\nu_1} \cdots t^{\nu_t}$ where $t=\max\{i,j \mid 1 \leq i, j \leq s\}$ and $\nu_d = \mu_d^{(2)} - \mu_d$, with $\mu_d^{(2)} = \sum_{i,j=d} (i,j)\mu_i\mu_j$, and $\mu_d=0$ if $d>s$. It follows that the cycle index is

$$(51) \quad \sum_{k=\mu_1+2\mu_2+\cdots+s\mu_s} (\mu_1!\, 2^{\mu_2} \cdots s^{\mu_s} \mu_s!)^{-1} x_1^{\nu_1} \cdots x_t^{\nu_t},$$

and the number in question is

$$(52) \quad \sum_{k=\mu_1+2\mu_2+\cdots+s\mu_s} (\mu_1!\, 2^{\mu_2} \cdots s^{\mu_s} \mu_s!)^{-1} 2^{\nu_1+\cdots+\nu_t}.$$

Of course if we let this number be D_k, then the number of binary systems B with finest decomposition $\{B_1, \cdots, B_k\}$ and $B_1=B_2=B_k$, where $|B|=n$ is given by:

$$(53) \quad \sum_{k|n} D_k B_I(n/k).$$

In order to handle the general situation, suppose that $D_{\bar{e}}=D_{(e_1, \cdots, e_t)}$ is the number of binary systems with the decomposition $\{B_1 \cdots, B_1, \cdots, B_t, \cdots, B_t\}$ (e_i copies of B_i, $B_1, \cdots, B_t$ distinct). We let the group $S_{e_1} \times \cdots \times S_{e_t}$ act on $((e_1+\cdots+e_t) \times (e_1+\cdots+e_t))_0$ via $(\phi_1 \cdots, \phi_t)(j)=\phi_i(j-(e_1+\cdots+e_{i-1}))+(e_1+\cdots+e_{i-1})$ if $e_1+\cdots+e_{i-1}<j \leq e_1+\cdots$

$+e_j$, and if $\phi=(\phi_1,\cdots,\phi_t)$, $\bar{\phi}(i,j)=(\phi(i),\ \phi(j))$. If ϕ_i has the type $1^{\mu_{i1}}\cdots s^{\mu_{is}}$, $\mu_{ij}\geq 0$, it follows that $\phi=(\phi_1,\cdots,\ \phi_t)$ has type $1^{\mu_1}\cdots s^{\mu_s}$ where $\mu_j=\sum_i \mu_{ij}$ and ϕ has the type described above formula (51).

The number of elements of $S_{e_1}\times\cdots\times S_{e_t}$ conjugate to ϕ is

(54) $\prod_{i=1}^{t}[(\mu_{i1}!\ 2^{\mu_{i2}}\cdots S^{\mu_{is}}\ \mu_{is}!)^{-1}\ e_i!]=\alpha(\mu_{ij})$

If we number the parts $B_1,\cdots,B_t$ continuously $B_1,\cdots,B_{e_1+\cdots e_t}$ and if we set $X(i,\ j)=1$ if $b_ib_j=b_i$, $X(i,j)=0$ if $b_ib_j=b_j$ for $b_r\in B_r$, then we have a colouration, and conversely every colouration determines a binary system. Now, isomorphisms move the parts of the decomposition around according to mappings $\bar{\phi}$, and thus if $P(S_{e_1}\times\cdots\times S_{e_t})$ denotes the cycle index, it follows that $D_{\vec{e}}=P(S_{e_1}\times\cdots\times S_{e_t};2,\cdots,2)$.
Note that the cycle index is given by

(55) $\sum_{e_i=1\mu_{i1}+\cdots+s\mu_{is},\,1\leq i\leq t}\alpha(\mu_{ij})/e_1!\ \cdots e_t!\ x_1^{\nu_1}\cdots x_t^{\nu_t}$

where $\nu_d=\mu_d^{(2)}-\mu_d$ as described above. For example if $e_1=\cdots=e_t=1$, then $D_{\vec{e}}=2^{t^2-t}$.

Suppose that $\vec{e}=(e_1,\cdots,e_1,\cdots,e_t,\cdots,e_t)$ where $e_1>\cdots>e_t$ and there are s_i copies of e_i. Let $\vec{N}=(N_{11},\cdots,N_{s_11},\cdots N_{s_tt})$ where $N_{1i}\geq N_{2i}\geq\cdots\geq N_{s_ii}$, $i=1,\ \cdots,\ t$. Write $B_I(\vec{N})=(B_I(N_{1i}),\cdots,B_I(Ns_tt))$ and $M*i=k$ if the number M appears k times in the vector $B_I(\vec{N})_i=(B_I(N_{1i}),\cdots,B_I(Ns_ii))$. Furthermore take

(56) $M*B_1(\vec{N})=\binom{B_I(M)}{M*1,\ M*2,\ \cdots,\ M*t,\ B_1(M)-\sum_{i=1}^{t}M*i}$

Finally, set $[B_I(\vec{N})]=\prod_{M=1}^{\alpha}\ \alpha*B_1(\vec{N})$ where α is sufficiently large (since then $\alpha*B_I(\vec{N})=1$).

Then, we claim that:

(57) $B(n)=\sum_{\vec{e}\cdot\vec{N}=n}[B_I(\vec{N})]D_{\vec{e}}$.

Since $D_{(1)}=1$, it follows that

(58) $B_I(n)=B(n)-\sum_{\vec{e}\cdot\vec{N}=n}[B_I(\vec{N})D_{\vec{e}}$.

which can be computed recursively. It only remains to verify formula (57). What we do is the following. Given any binary system B with $|B|=n$ it has a cover $\{B_1,\cdots,B_1,\cdots,B_t,\cdots,B_t\}$ where there are e_i copies of B_i. We may sort these according to number $(e_1 \geq e_2 \geq \cdots \geq e_t)$ and within a fixed number of copies e_i of which there are several we order according to size $(N_{1i} \geq N_{2i} \geq \cdots \geq N s_i i)$, i.e., we are working with the usual cascading vectors. In effect, if we apply this system we have used up all available freedom and the method now follows the usual rules. Thus $\vec{e} \cdot \vec{N} = n$, since we are taking the union of subsets. Also, in each slot we have a certain number of possible choices, viz., $B_I(N_{ij})$ in the slot corresponding to N_{ij}. Next, if we have $M*i$ components in the decomposition which occur with the same frequency and which have the same number of elements, then we have $(M*i)!$ equivalent arrangements. Thus, since the total number of times we have to select different components with M elements is $\sum_{i=1}^{t} M*i$, the total contribution to the possibilities for the type of arrangement we have described consists of the product of the two coefficients

$$(59) \quad \binom{B_1(M)}{\sum_{i=1}^{t} M*i} \binom{\sum_{i=1}^{t} M*i}{M*1, \cdots, M*t} = M*B_I(\vec{N}).$$

Indeed, the first coefficient counts the number of selections, while the second coefficient counts the number of arrangements. Finally, since binary systems with different numbers of elements are sowieso not isomorphic, it follows that with the vectors $\vec{e}$ and $\vec{N}$ fixed, that the total count of possibilities is simply the product. Thus $B_I(\vec{N})$ counts the number of distinct decompositions, which then needs to be multiplied by $D_{\vec{e}}$ to give the number of ways they can be put together. Summing over all proper bipartitions yields formula (57) and we're done.

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