Kyungpook Math. J.
Volume 16, Number 2
December, 1976

FINITELY NEIGHBORABLE GROUPOIDS

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Let Gpd be the category of compact groupoids and continuous homomorphisms. A subclass S is constructed by taking Cartesian products of finite groupoids, closed subgroupoids of the products and continuous homomorphic images thereof. It will be shown that S is precisely the class of finitely neighborable groupoids. Since a compact connected group like the circle group is not finitely neighborable, S is a proper subclass of Gpd.

A groupoid S is a Hausdorff space with a binary operation $\cdot : S\times S\rightarrow S$. A groupoid S is said to be finitely neighborable if for every open cover of S, there exists a finite refinement $\mathscr{U}$ such that $S=\bigcup_{U\in\mathscr{U}} U^0$ where U^0 is the interior of U and if $U, V\in\mathscr{U}$, then there is a $W\in\mathscr{U}$ such that $UV\subset W$. Clearly, a finitely neighborable groupoid is compact.

LEMMA 1. *A finitely neighborable groupoid is a topological groupoid* (*i.e.*, *the operation is jointly continuous*).

PROOF. Let $x, y\in S$ where S is a finitely neighborable groupoid, and let $xy\in U$ where U is an open set. Then $\mathscr{V}=\{U, S\backslash xy\}$ is an open cover of S. Let $\mathscr{U}$ be the refinement, and $x\in A^0$, $y\in B^0$ for some $A, B\in\mathscr{U}$. Then $AB\subset C$ for some $C\in\mathscr{U}$. Hence $C\subset U$ since $xy\in C$.

In [2], the following theorem was shown for semigroups, but the proof could be adapted to the groupoid case.

THEOREM 2. *Let* $\{S_\alpha\}$ *be a collection of finite groupoids. Then each of the following is finitely neighborable*:

(1) *a Cartesian product of* $\{S_\alpha\}$

(2) *a closed subgroupoid of the product*

(3) *a continuous homomorphic image of a closed subgroupoid of the product.*

If one takes all finite groupoids and constructs the class given by (1), (2), (3), then the resulting class S is contained in the class of finitely neighborable groupoids. The ensuing propositions establish that the two classes are the same.

Before proceeding to the lemmas, a nerve $\mathscr{N}$ of a groupoid S is defined to be a finite cover of S such that (1) $S=\bigcup_{N\in\mathscr{N}} N^0$ and (2) if $U, V\in\mathscr{N}$, then there exists a unique $W\in\mathscr{N}$ such that $UV\subset W$ and if $UV\subset W'$ where $W'\in\mathscr{N}$, then $W\subset W'$ and (3) $\mathscr{N}$ is closed under nonempty intersection. If $\mathscr{N}$ is a nerve, one can define an operation on the finite set $\mathscr{N}$ by $U*V$ to be the unique W given by the nerve. If (S, d) is a metric space with a bounded metric d, then let $b(A)=\sup[\beta>0: N_\beta(x)\subset A]$ where A is a subset of S and $N_\beta(x)$ is the open sphere of x with radius β.

LEMMA 3. *If S is a finitely neighborable groupoid, then every open cover has a nerve refinement.*

PROOF. Let $\mathscr{U}$ be an open cover of S. Then there exists a finite cover $\mathscr{V}$ of S given by the definition. Let $\mathscr{W}$ be the collection of all nonempty intersections of $\mathscr{V}$. Then $\mathscr{W}$ refines $\mathscr{U}$ since $\mathscr{V}$ is contained in $\mathscr{W}$. Let $A, B\in\mathscr{W}$. Then $A=A_1\cap\dots\cap A_n$ and $B=B_1\cap\dots\cap B_m$ where A_i, $B_i\in\mathscr{V}$. There is a $C\in\mathscr{V}$ such that $A_1B_1\subset C$. So $AB\subset C$. Let D be the intersection of all sets in $\mathscr{V}$ which contain AB. Then D is that unique element in $\mathscr{W}$ that contains AB. Also $S=\bigcup_{V\in\mathscr{V}} V^0\subset\bigcup_{W\in\mathscr{W}} W^0$. Hence $\mathscr{W}$ is the desired refinement.

If $\mathscr{C}$ is a nerve of a metric finitely neighborable groupoid S and $i\in N$ (N is the set of positive integers), then we denote $\mathscr{U}\underset{i}{<}\mathscr{C}$ if $\mathscr{U}$ is a nerve of S satisfying:

(a) $\operatorname{diam} U<1/2^i$ for each $U\in\mathscr{U}$

(b) $\mathscr{U}$ refines $\{C^0 \mid C\in\mathscr{C}\}$

(c) if $U\in\mathscr{U}$, $W\in\mathscr{C}$ and $U\cap W\neq\phi$, then $U\cap W\in\mathscr{U}$

(d) if $U\in\mathscr{U}, W\in\mathscr{C}$ and $W^0\neq\phi$, then $\operatorname{diam} U<\frac{1}{2}\, b(W)$.

LEMMA 4. *If $\mathscr{C}$ is a nerve of a metric finitely neighborable groupoid S and $i\in N$, then there exists $\mathscr{U}\underset{i}{<}\mathscr{C}$.*

PROOF. Let L be a Lebesgue number for $\{C^0\neq\phi \mid C\in\mathscr{C}\}$. Choose $\varepsilon>0$ such that ε is smaller than all of L, $1/2^i$ and $(1/2)b(W)$ for all $W\in\mathscr{C}$. By Lemma 3, there exists $\mathscr{W}$ a nerve of S with diam $U\leq\varepsilon$ for each $U\in\mathscr{U}$. Then $\mathscr{U}=\{W\cap C\neq\phi \mid W\in\mathscr{W},\ C\in\mathscr{C}\}$ is the desired nerve.

THEOREM 5. *If S is a metric finitely neighborable groupoid, then there exists* *a continuous homomorphism f from a compact totally disconnected groupoid P onto S.*

PROOF. By Lemma 4, one can get a sequence of nerves $\mathscr{U}_1, \mathscr{U}_2, \ldots$ such that $\mathscr{U}_{i+1} \underset{i+1}{<} \mathscr{U}_i$. Eor each i, consider $\mathscr{U}_i$ as a finite groupoid with the multiplication $U{*}V$.

Let $P = \{(U_i)_{i\in N} \in \prod_{i\in N} \mathscr{U}_i | U_{i+1} \subset U_i$ for each $i \in N\}$.

One proceeds to show that P is a closed subgroupoid of the product space. Suppose $(U_i)_{i\in N} \notin P$. Then there exists $j \in N$ such that $U_j \not\subset U_{j-1}$. Let $W = \{U_1\} \times \{U_2\} \times \ldots \times \{U_j\} \times \prod_{i>j} \mathscr{U}_i$. Then $(U_i)_{i\in N} \in W$ and $P \cap W = \phi$. Hence P is closed.

Suppose $(U_i)_{i\in N}$, $(V_i)_{i\in N} \in P$. Let $(U_i){*}(V_i) = (U_i{*}V_i) = (W_i)$. If $j \in N$, then $U_{j+1} \subset U_j$ and $U_{j+1} \subset V_j$. Hence $U_{j+1} V_{j+1} \subset U_j V_j \subset W_j$. Since $U_{j+1}\, V_{j+1} \subset W_{j+1}$ and $W_{j+1} \cap W_j \in \mathscr{U}_{j+1}$ by property (c), then $W_{j+1} \subset W_{j+1} \cap W_j$ since $\mathscr{U}_{j+1}$ is a nerve. Thus $W_{j+1} \subset W_j$ implies $(W_i) \in P$.

If $(U_i)_{i\in N} \in P$, then $\bigcap_{i\in N} \overline{U}_i$ contains exactly one point since diam U_i converges to 0. Define $f; P \to S$ by $f((U_i)_{i\in N}) = p$ where $p \in \bigcap_{i\in N} \overline{U}_i$. Suppose $f((U_i)_{i\in N}) = p \in W$ where W is open in S. Then there exists U_j for some j such that $p \in \overline{U}_j \subset W$. Hence $f(\{U_1\} \times \cdots \times \{U_j\} \times \prod_{i>j} \mathscr{U}_i \cap P) \subset W$ which implies that f is continuous.

Suppose $(U_i)_{i\in N}$, $(V_i)_{i\in N} \in P$ and $(U_i){*}(V_i) = (W_i)$ and $f((U_i)_{i\in N}) = p$, $f((V_i)_{i\in N}) = q$. Then $pq \in (\cap \overline{U}_i)(\cap \overline{V}_i) \subset \overline{U_i V_i} \subset \cap \overline{W}_i$ since $U_i V_i \subset W_i$. But $\cap \overline{W}_i$ contains the unique pcint $f((W_i)_{i\in N})$. Hence $f((W_i)_{i\in N}) = pq = f((U_i))\, f((V_i))$, i.e., f is a homomorphism.

Finally, we have to show f is onto. Suppose there exists $x \in S \backslash f(P)$. Then there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \cap f(P) = \phi$. Let $n \in N$ such that $0 < 1/2^n < \varepsilon/2$. Then $x \in U_n^0$ for some $U_n \in \mathscr{U}_n$. Since diam $U_n < 1/2^n$ then $U_n \cap f(P) = \phi$. Choose $r > 0$ such that $(1/2)b(U_n) < r < b(U_n)$. Then by the definition of $b(U_n)$, there exists $N_r(y) \subset U_n$. Let $y \in U_{n+1}^0$ for some $U_{n+1} \in \mathscr{U}_{n+1}$. One proceeds to show that $U_{n+1} \subset U_n$. Let $z \in U_{n+1}$. Then $d(y, z) \leq$ diam $U_{n+1} < (1/2)b(U_n) < r$. Hence $z \in N_r(y) \subset U_n$. So we have found $U_{n+1} \in \mathscr{U}_{n+1}$ such that $U_{n+1} \subset U_n$. Continuing this process, one can find U_{n+1}, U_{n+2}, $\ldots$ such that $U_{n+i} \in \mathscr{U}_{n+i}$ and $U_{n+i} \subset U_{n+i-1}$.

Also $U_n \subset U_{n-1}$ for some $U_{n-1} \in \mathscr{U}_{n-1}$ since $\mathscr{U}_n$ refines $\mathscr{U}_{n-1}$. Hence we have

$U_n \subset U_{n-1} \subset U_{n-2} \subset \cdots \subset U_2 \subset U_1$ where $U_i \in \mathscr{U}_i$. Thus there exists a sequence U_1, U_2, U_3, $\cdots U_n$, $\cdots$ such that $U_{i+1} \subset U_i$ and $U_i \in \mathscr{U}_i$. Let $a=f((U_i)_{i\in N})$. Then $a \in \bigcap_{i\in N} \overline{U}_i \subset \overline{U}_n$. But $\overline{U}_n \cap f(P)=\phi$. Thus we have a contradiction.

THEOROM 6. *If S is a finitely neighborable groupoid, then S is a quotient of a compact totally disconnected groupoid.*

PROOF. Since S is compact, then $S=\varprojlim S_\alpha$ where S_α is a compact metrizable groupoid and the projection function π_α from S to S_α is surjective. Hence S_α is finitely neighborable for each α. Let f_α be a continuous homomorphism from a compact totally disconnected groupoid T_α onto S_α. Then f: $\Pi T_\alpha \to \Pi S_\alpha$ is a continuous surjective homomorphism where $f((t_\alpha))=(f_\alpha(t_\alpha))$. Then $f|_{f^{-1}(S)}$ is a continuous homomorphism from $f^{-1}(S)$ onto S. Since $f^{-1}(S)$ is a subspace of a product of totally disconnected spaces T, then $f^{-1}(S)$ is totally disconnected.

COROLLARY 7. *If X is a compact Hausdorff space, then there is a compact totally disconnected Hausdorff space T and a continuous function from T onto X.*

PROOF. Put the multiplication $xy=x$ on X and apply Theorem 6.

QUESTION. If S is a finitely neighborable semigroup, then is there a compact totally disconnected semigroup T and a cotinuous homomorphism from T onto S?

These are some examples of finitely neighborable semigroups.

EXAMPLE 1. If S is a compact semigroup with a basis of open subsemigroups, then S is finitely neighborable.

More specific examples like compact semilattices with small semilattices can be found in [1], [3], [4] and [5], e.g., $[0,1]$ with $xy=\min\{x,y\}$ or a product of $[0,1]$ or a closed subsemigroup thereof.

EXAMPLE 2. The interval $[0,t]$ under real multipilication where $t<1$.

Since h: $[0,t]\to[0,1/2]$ defined by $h(x)=(1/2)^{\log x/\log t}$ is a topological isomorphism, it is enough to check $[0,1/2]$ is finitely neighborable. Use these facts about $[0, 1/2]$: (1) if $|x-y|<h$ and $|a-b|<k$, then $|xa-yb[<\max\{h,k\}$, (2) if $\varepsilon>0$, then there exists $n>0$ such that $[0,1/2]^n\subset[0,\varepsilon]$. Let $\mathscr{U}$ be a finite cover of $[\varepsilon,1/2]$ by open sets of diameter less than ε. Form $\mathscr{V}$ by taking $U_1U_2\cdots U_k$ where $k\leq n$ and $U_i\in\mathscr{U}$. Then $\mathscr{V}\cup\{[0,\varepsilon)\}$ is the required refinement.

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