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UNIONS OF FIRST CATEGORY SPACES

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The Banach Category Theorem [6] says that the union of spaces of first category, each open in the union, is a space of first category. We give an example showing that the "open" requirement in the Banach Category Theorem cannot be replaced by "closed" even if the union is a separable metric space(each space being of first category in itself). The first two theorems in this paper present the situation regarding unions of first category spaces, second category spaces, and Baire spaces.

A space is of *first category* provided that it can be written as a countable union of nowhere dense subsets(i. e., subsets whose closures have empty interiors). Note that when we say that a subset of a space is of first category we mean that it is of first category (in itself) as a subspace. A space which is not of first category is of *second category*. Finally, a space having every open subset of second category is called a *Baire space*.

The property of being a Baire space and a number of other properties, such as being a pseudo-complete space (see [3] and [5]) and being a weakly α-favorable space (see [7]), have certain theorems which they all share. The following are among these.

(A) every open subspace of a space with property P has property P.

(B) If every point of a space has an open neighborhood having property P, then the space has property P.

(C) If a dense subspace of a space has property P, then the space has property P.

Certain conclusions can be obtained directly from these statements, such as the following lemma.

LEMMA 1. *If property P satisfies statements* (A), (B), *and* (C), *then a finite union of spaces having property P has property P.*

PROOF. Suppose that $X=Y\cup Z$, where Y and Z have property P. Now $X\backslash\overline{Y}$ and $X\backslash\overline{Z}$ are open in Z and Y, respectively, so that they have property P by statement (A). Let $W=(X\backslash\overline{Y})\cup(X\backslash\overline{Z})$. Now $X\backslash\overline{W}\subset\overline{Y}$, and $\overline{Y}$ has property P by statement (C). Thus $X\backslash\overline{W}$ has property P by statement (A) again. Therefore

$W\cup(X\setminus\overline{W})$ has property P by statement (B). But then X has property P by statement (C) again, since $W\cup(X\setminus\overline{W})$ is dense in X.

The next lemma and its proof, which is straightforward, can be found in [4].

LEMMA 2. *If $\mathscr{N}$ is a family of nowhere dense subsets of space X which is locally finite at a dense set of points of X, then $\cup\mathscr{N}$ is nowhere dense in X.*

We now give the situation for locally finite unions of first or second category spaces. In the following two theorems, proofs involving Baire spaces use only the properties in statements (A), (B), and (C). For this reason, in part (ii) of each of these theorems, the words "Bair space" can be replaced by the name of any other property satisfying the statements (A), (B), and(C).

THEOREM 1. *Let $\{X_\alpha\}$ be a cover of X which is locally finite at a dense set of points of X.*

(i) *If each X_α is of first category, then X is of first category.*

(ii) *If each X_α is a Baire space, then X is a Baire space.*

PROOF. (i) Each $X_\alpha=\bigcup_{i=1}^{\infty} N_\alpha^i$, where each N_α^i is a nowhere dense subset of X_α, and hence is nowhere dense in X. So for each i, $\{N_\alpha^i\}$ is locally finite at a dense set of points of X. Let $N^i=\bigcup_\alpha N_\alpha^i$, which is nowhere dense in X by Lemma 2. Also $X=\bigcup_\alpha X_\alpha=\bigcup_\alpha(\bigcup_{i=1}^{\infty} N_\alpha^i)=\bigcup_{i=1}^{\infty} N^i$, so that X is of first category.

(ii) Let U be a nonempty open subset of X. Then U contains a nonempty open subset V intersecting only finitely many members of $\{X_\alpha\}$, say $X_{\alpha_1},\dots,X_{\alpha_n}$. Now $\bigcup_{i=1}^{n} X_{\alpha_i}$ is a Baire space by Lemma 1. Also V is an open subset of $\bigcup_{i=1}^{n} X_{\alpha_i}$, so that V is a Baire space by statement (A). Since U is arbitrary, X is a Baire space by statements (B) and (C).

On the other hand, the union of two second category spaces may be of first category, as seen by the following example. Let $X=Q_1\cup Q_2\cup\{p_1,p_2\}$, where Q_1 and Q_2 are disjoint copies of the rationals and p_1 and p_2 are distinct points not in Q_1 or Q_2. Let basic neighborhoods of points in Q_1 and Q_2 be the usual open sets in Q_1 and Q_2, respectively. Let a basic neighborhood of p_1 be $\{p_1\}$ union a right ray in Q_1, and let a basic neighborhood of p_2 be $\{p_2\}$ union a right ray in Q_2. Finally let $X_1=Q_1\cup\{p_2\}$ and $X_2=Q_2\cup\{p_1\}$. Then X is a first category space is the union of two second category subsets X_1 and X_2.

Clearly the countable union of first category spaces is of first category. However, it is easy to see that if the local finiteness of $\{X_\alpha\}$ in Theorem 1 is replaced by $\{X_\alpha\}$ being countable, then part (ii) of the theorem becomes false. Thus some additional hypothesis is needed for arbitrary unions. The natural such hypothesis is that each X_α be open in X, as the following theorem illustrates.

THEOREM 2. *Let $\{X_\alpha\}$ be a family of open subsets of X whose union is dense in X.*

(i) *X is of first category if and only if every X_α is of first category.*

(ii) *X is a Baire space if and only if every X_α is a Baire space.*

PROOF. Part (i) is true because every open subspace of a space of first category is of first category and because of the Banach Category Theorem stated in the introductory paragraph above. Part (ii) follows from statements (A), (B), and (C).

A natural question now is whether the hypothesis in Theorem 2 requiring each X_α to be "open" in X can be replaced by some other condition such as "closed" or "G_δ-subset". This is obviously false for part (ii), but is a more difficult question for part (i). An example of a homogeneous pseudo-complete space X having a closed subspace which is of first category is R^c, where c denotes $2^{\aleph_0}$ and where R is the reals with the usual topology. The proof that R^c contains a closed copy of the rationals can be found in [1], so that R^c can be written as the union of closed subsets each of which is of first category. Note that these closed subsets are necessarily not G_δ-subsets. In the following theorem, such an example is constructed where the closed first category subsets are also G_δ-subsets.

THEOREM 3. *Assuming the continuum hypothesis or Martin's axiom, there exists a separable metrizable Baire space which has a decomposition into closed subspaces each of which is of first category (in itself).*

PROOF. Let X be the real numbers with the usual topology. Let $\mathcal{E}$ be the set of equivalence classes of the relation on X defined by: x is equivalent to y if and only if $x-y$ is rational. Let $\{x_\alpha | \alpha < c\}$ be a well-ordering of X, and let $\{G_\alpha | \alpha < c\}$ be a well-ordering of the somewhere dense G_δ-subsets of $X \times X$. Let π_1 and π_2 be the projection maps from $X \times X$ onto the first and second factors, respectively.

Let p_0 be the first x_α contained in $\pi_1(G_0)$, let q_0 be the first x_α contained in

$\pi_2[\pi_1^{-1}(p_0)\cap G_0]$, and let $X_0\in\mathcal{E}$ such that $p_0\in X_0$. Now suppose that for $0<\gamma<c$ and for each $\beta<\gamma$, $(p_\beta, q_\beta)\in G_\beta\setminus([\bigcup_{\delta<\beta}(X_\delta\times X)]\cup[\bigcup_{\delta<\beta}(X\times\{q_\delta\})])$(or $(p_0, q_0)\in G_0$ if $\beta=0$) and $p_\beta\in X_\beta\in\mathcal{E}$. Then define p_γ, q_γ, and X_γ as follows. Let $F_\gamma=[\bigcup_{\beta<\gamma}(X_\beta\times X)]\cup[\bigcup_{\beta<\gamma}(X\times\{q_\beta\})]$, which because of the continuum hypothesis or Martin's axiom is a first category subset of $X\times X$. Thus $G_\gamma\setminus F_\gamma$ is nonempty. Define p_γ to be the first x_α contained in $\pi_1(G_\gamma\setminus F_\gamma)$, q_γ to be the first x_α contained in $\pi_2[\pi_1^{-1}(p_\gamma)\cap(G_\gamma\setminus F_\gamma)]$, and $X_\gamma\in\mathcal{E}$ such $p_\gamma\in X_\gamma$. Thus $\{X_\alpha|\alpha<c\}$, $\{p_\alpha|\alpha<c\}$, and $\{q_\alpha|\alpha<c\}$ are defined by transfinite induction. Note that for $\alpha\neq\beta$, $q_\alpha\neq q_\beta$. Finally, let $Z=\bigcup_{\alpha<c}(X_\alpha\times\{q_\alpha\})$, which is a dense subspace of $X\times X$ and in fact is a Baire space since every somewhere dense G_δ-subset of $X\times X$ intersects Z (see [2] or [4]). Clearly each $X_\alpha\times\{q_\alpha\}$ is closed in Z and is first category in itself since it is homeomorphic to the rationals.

This example also shows that the converse to the following theorem found in [4] is not true.

THEOREM 4. *Let f be a continuous open function from a space X having a countable pseudo-base onto a Baire space Y. If $Y\setminus\{y\in Y|f^{-1}(y)$ is a Baire space$\}$ is of first category, then X is a Baire space.*

The projection π_1 is continuous and open, and $\pi_1(Z)$ can easily be seen to be a Baire space. However $\pi_1^{-1}(y)$ is of first category for every $y\in\pi_1(Z)$.

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