

GENERAL INTEGRALS INVOLVING PRODUCTS OF A
 GENERALIZED HYPERGEOMETRIC POLYNOMIALS
 AND H -FUNCTION

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1. Introduction

Several integrals involving the product of a polynomial and a function have been evaluated from time to time. Saxena [5] has given an integral involving the product of Jacobi polynomial and Meijer's G -function. From this integral by giving particular values to the parameters in G -function he has obtained a number of integrals involving the product of Jacobi polynomials and a function. Sharma [6] has given integrals involving the product of a Gauss's hypergeometric function and a G -function. I have evaluated two very general integrals involving the product of the generalized hypergeometric polynomials and H -functions by expressing the H -function as Mellin-Barnes integral and interchanging the order of the integrations, I have shown that from these integrals not only the integrals given by Saxena [5] and Sharma [6] are deducible as special cases, but also a large number of interesting results can be deduced.

We shall be using the Barnes contour integral for the H -function, Braaksma [2 p.239] viz.

$$(1.1) \quad H_{p,q}^{m,n} \left[x \begin{matrix} (a_1, \mu_1), \dots, (ap, \mu p) \\ (b_1, \eta_1), \dots, (bq, \eta q) \end{matrix} \right] \\ = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - \eta_j s) \prod_{j=1}^n \Gamma(1 - aj + \mu_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \eta_j s) \prod_{j=n+1}^p \Gamma(a_j - \mu_j s)} x^s ds$$

where L is a suitable contour as defined in Braaksma [2 p.239]

From the definition of G -function Erdelyi [3 p.207] it follows that

$$H_{p,q}^{m,n} \left[x \begin{matrix} (a_1, 1), \dots, (a_p, 1) \\ (b_1, 1), \dots, (b_q, 1) \end{matrix} \right] = G_{p,q}^{m,n} \left(x \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right)$$

The generalized hypergeometric polynomials is defined Jain [4 p.177] as

$$(1.3) \quad f_n^{(c,k)} [(ap), (bq); x] = f_n^{(c,k)} (a_1, \dots, a_p, b_1, \dots, b_q; x)$$

$$= \frac{(c)_n}{n!} {}_{p+k}F_{q+k} \left[\begin{matrix} -n, \Delta(k-1, c+n), (ap), \\ \Delta(k, c); (bq) \end{matrix} \middle| (k-1)^{k-1} x \right]$$

where n and k are non negative integers and

$$\Delta(k, c) = \frac{c}{k}; \quad \frac{c+1}{k}, \dots, \frac{c+k-1}{k}.$$

When $c=1$, $k=2$, (1.3) reduces to sister celines polynomials also. Jain [4 p. 177, (1.2), (1.3)]

$$(1.4) \quad f_n^{(1+a, 1)}(x) = L_n^{(a)}(x) \text{ (Laguerre polynomials.)}$$

and

$$(1.5) \quad f_n^{(1+a+b, 2)} \left(\frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b, 1 + \frac{a}{2} + \frac{b}{2}; 1+a; x \right) \\ = \frac{(1+a+b)_n}{(1+a)_n} P_n^{(a, b)}(1-2x)$$

(Jacobi polynomials).

For m a positive integer, the multiplication formula for Gamma functions is

$$(1.6) \quad \Gamma(mz) = (2\pi)^{1/2(1-m)} (m)^{mz-1/2} \prod_{\lambda=0}^{m-1} \Gamma(z + \lambda/m).$$

The symbol (ap) stands for the p parameters

a_1, a_2, \dots, a_p and

$$\Delta(k, (ap)) = \Delta(k, a_1), \Delta(k, a_2), \dots, \Delta(k, a_p).$$

It may be noted that the parameters and the variables are such that the functions involved exist and the integrals are convergent.

2. Integral (I)

The first integral to be established is

$$(2.1) \quad \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} f_n^{(c, k)} [(ap), (bq), t] H_{p, q}^{u, v} \left[t^l x \middle| \begin{matrix} (dp), (\mu p) \\ (eq), (\eta q) \end{matrix} \right] dt \\ = \frac{(c)_n \Gamma(\beta)}{n! (l)^\beta} \\ \times \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^u \Gamma(e_j - \eta_j s) \prod_{j=1}^v \Gamma(1 - d_j + \mu_j s) \prod_{\lambda=0}^{l-1} \Gamma\left(-\frac{\alpha + \lambda}{l} + \mu s\right)}{\prod_{j=u+1}^q \Gamma(1 - e_j + \eta_j s) \prod_{j=v+1}^p \Gamma(d_j - \mu_j s) \prod_{\lambda=0}^{l-1} \Gamma\left(\frac{\alpha + \beta + \lambda}{l} + \mu s\right)} ds$$

$$\times {}_{p+k+1}F_{q+k+1} \left[\begin{matrix} -n, \Delta(k-1, c+n), (ap), \alpha + \mu ls; \\ \Delta(k, c); (bq); \alpha + \beta + \mu ls \end{matrix} \right]_{(k-1)^{k-1}} x^s ds$$

$$\operatorname{Re}(\alpha) > 0; \operatorname{Re}(\beta) > 0.$$

The integral is valid under the following sets of conditions.

(i) $q \geq 1; p < q$ or when $p = q$ then $|x| < 1$

and $\operatorname{Re}(\alpha + l e_j) > 0$ for $j = 1, 2, \dots, u$.

(ii) $p \geq q, 2(u+v) \geq p+q; |\arg z| < (u+v - \frac{1}{2}p - \frac{1}{2}q)\pi$

$\operatorname{Re}(\alpha + l e_j) > 0; \text{ for } j = 1, 2 \dots u.$

$$\operatorname{Re} \left[l \left(\sum_1^p d_j - \sum_1^q c_j - 1/2 \right) + (p+q)(\alpha - 1/2) \right] > -1$$

(iii) $p > q; v = 0; 2u > p+q$ and $|\arg z| < (u - 1/2p - 1/2q)\pi.$

The contour L taken here is a straight line along the imaginary axis extending from $-i\infty$ to $+i\infty$ with indentations (if necessary), so that the poles of $\Gamma(e_j - \eta_j s)$ $j = 1, 2 \dots u$ and that of the functions of this type are to the right of the contour L and the poles of $\Gamma(1 - d_j + \mu_j s)$, $j = 1, 2 \dots v$ and that of the functions of this type are to the left of it.

PROOF. Substituting the value of H -function and $f_n^{(c,k)} [(ap), (bq); t]$ from (1.1) and (1.3) respectively in the left hand side of (2.1) and interchanging the order of integrations which is permissible under the conditions of (2.1) we have,

$$\begin{aligned} & \frac{(C)_n}{n!} \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^u \Gamma(e_j - \eta_j s) \prod_{j=1}^v \Gamma(1 - d_j + \mu_j s)}{\prod_{j=u+1}^q \Gamma(1 - e_j + \eta_j s) \prod_{j=v+1}^p \Gamma(d_j - \mu_j s)} \\ & \times \sum_{r=0}^n \frac{(-n)_r \prod_{\lambda=0}^{k-2} \left[\frac{c+n+\lambda}{k-1} \right]_r (ap)_r (k-1)^{(k-1)r}}{r! \prod_{\lambda=0}^{k-1} \left[\frac{c+\lambda}{k} \right]_r (bq)_r} \\ & \times \int_0^1 t^{\alpha+r+l\mu s-1} (1-t)^{\beta-1} dt x^s ds. \end{aligned}$$

On using the Beta Integral [3 p.9] and the multiplication formula (1.6) we get the right hand side of (2.1). Thus (2.1) is established.

(b) Integral II

Proceeding on the same lines and using the Euler's integral [3 p.1] we can get the following integral. For $\text{Re}(\alpha) > 0$

$$\begin{aligned}
 (2.2) \quad & \int_0^{\infty} e^{-t} t^{\alpha-1} f_n^{(c,k)}[(ap), (bq); t] H_{p,q}^{u,v} \left[t^{l\mu} x \left| \begin{matrix} ((dp), (\mu p)) \\ ((eq), (\eta q)) \end{matrix} \right. \right] dt \\
 & = \frac{(c)_n (2\pi)^{1/2(1-l)} (l)^{\alpha-1/2}}{n!} \\
 & \times \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^u \Gamma(e_j - \eta_j s) \prod_{j=1}^v \Gamma(1 - d_j + \mu_j s) \prod_{\lambda=0}^{l-1} \Gamma\left(\frac{\alpha + \lambda}{l} + \mu s\right)}{\prod_{j=u+1}^q \Gamma(1 - e_j + \eta_j s) \prod_{j=v+1}^p \Gamma(d_j - \mu_j s)} \\
 & \times {}_{p+k+1}F_{q+k+1} \left[\begin{matrix} -n, \Delta(k-1, c+n), (ap); \alpha + \mu l s; \\ \Delta(k, c), (bq) \end{matrix} \right] (k-1)^{k-1} (l^{l\mu} x)^s ds
 \end{aligned}$$

under the conditions of (2.1).

(c) A special case of integral I

If we substitute $t = \sin^2 \theta$ in (2.1), we get the following integral.

$$\begin{aligned}
 (2.3) \quad & \int_0^{\pi/2} \sin^{2\alpha-1} \theta \cos^{2\beta-1} \theta f_n^{(c,k)}[(ap), (bq); \sin^2 \theta] H_{p,q}^{u,v} \left[x \sin^{2l\mu} \theta \left| \begin{matrix} ((dp), (\mu p)) \\ ((eq), (\eta q)) \end{matrix} \right. \right] d\theta \\
 & = \frac{1}{2} \frac{(c)_n \Gamma(\beta)}{n! (l)^\beta} \\
 & \times \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^u \Gamma(e_j - \eta_j s) \prod_{j=1}^v \Gamma(1 - d_j + \mu_j s) \prod_{\lambda=0}^{l-1} \Gamma\left(\frac{\alpha + \lambda}{l} + \mu s\right)}{\prod_{j=u+1}^q \Gamma(1 - e_j + \eta_j s) \prod_{j=v+1}^p \Gamma(d_j - \mu_j s) \prod_{\lambda=0}^{l-1} \Gamma\left(\frac{\alpha + \beta + \lambda}{l} + \mu s\right)} \\
 & \times {}_{p+k+1}F_{q+k+1} \left[\begin{matrix} -n, \Delta(k-1, c+n); (ap); \alpha + \mu l s; \\ \Delta(k, c), (bq); \alpha + \beta + \mu l s; \end{matrix} \right] (k-1)^{k-1} x^s ds.
 \end{aligned}$$

3. The general integrals involving the product of a generalized hypergeometric polynomials and Meijer's G-function

If we take $\mu_j = \eta_j = \mu = 1$, $j = 1, 2, \dots$, then by virtue of (1.2) the integrals (2.1), (2.2) and (2.3) reduce to the following general integrals involving the product of the generalized hypergeometric polynomials and Meijer's G-function. The conditions of (2.1) still hold.

Integral I:

$$\begin{aligned}
 (3.1) \quad & \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} f_n^{(c,k)} [(ap), (bq); t] G_{p,q}^{u,v} \left(t^l x \begin{matrix} (dp) \\ (eq) \end{matrix} \right) dt \\
 &= \frac{(c)_n \Gamma(\beta)}{n! (l)^\beta} \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^u \Gamma(e_j - s) \prod_{j=1}^v \Gamma(1 - d_j + s) \prod_{\lambda=0}^{l-1} \Gamma\left(\frac{\alpha + \lambda}{l} + s\right)}{\prod_{j=u+1}^q \Gamma(1 - e_j + s) \prod_{j=v+1}^p \Gamma(d_j - s) \prod_{\lambda=0}^{l-1} \Gamma\left(\frac{\alpha + \beta + \lambda}{l} + s\right)} \\
 & \times {}_{p+k+1}F_{q+k+1} \left[\begin{matrix} -n, \Delta(k-1, c+n); (ap), (\alpha + ls); \\ \Delta(k, c); (bq); (\alpha + \beta + ls); \end{matrix} \right. \left. (k-1)^{k-1} \right] x^s ds.
 \end{aligned}$$

Integral II:

$$\begin{aligned}
 (3.2) \quad & \int_0^\infty e^{-t} t^{\alpha-1} f_n^{(c,k)} [(ap), (bq), t] G_{p,q}^{u,v} \left(t^l x \begin{matrix} (dp) \\ (eq) \end{matrix} \right) dt \\
 &= \frac{(c)_n (2\pi)^{1/2(1-l)} (l)^{\alpha-1/2}}{n!} \\
 & \times \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^u \Gamma(e_j - s) \prod_{j=1}^v \Gamma(1 - d_j + s) \prod_{\lambda=0}^{l-1} \Gamma\left(\frac{\alpha + \lambda}{l} + s\right)}{\prod_{j=u+1}^q \Gamma(1 - e_j + s) \prod_{j=v+1}^p \Gamma(d_j - s)} \\
 & \times {}_{p+k+1}F_{q+k+1} \left[\begin{matrix} -n, \Delta(k-1, c+n); (ap); (\alpha + ls) \\ \Delta(k, c); (bq); \end{matrix} \right. \left. (k-1)^{k-1} \right] (l^l x)^s ds.
 \end{aligned}$$

Integral III:

$$(3.3) \quad \int_0^{\pi/2} \sin^{2\alpha-1} \theta \cos^{2\beta-1} \theta f_n^{(c,k)} [(ap), (bq); \sin^2 \theta] G_{p,q}^{u,v} \left(x \sin^{2l} \theta \begin{matrix} (dp) \\ (eq) \end{matrix} \right) d\theta$$

$$= \frac{(c)^n \Gamma(\beta)}{2(n)! (l)^\beta} \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^u \Gamma(e_j - s) \prod_{j=1}^v \Gamma(1 - d_j + s) \prod_{\lambda=0}^{l-1} \Gamma\left(\frac{\alpha + \lambda}{l} + s\right)}{\prod_{j=u+1}^q \Gamma(1 - e_j + s) \prod_{j=v+1}^p \Gamma(d_j - s) \prod_{\lambda=0}^{l-1} \Gamma\left(\frac{\alpha + \beta + \lambda}{l} + s\right)} \\ \times {}_{p+k+1}F_{q+k+1} \left[\begin{matrix} -n, \Delta(k-1, c+n); (ap); (\alpha + ls); \\ \Delta(k, c); (bq); (\alpha + \beta + ls); \end{matrix} ; (k-1)^{k-1} \right] x^s ds.$$

4. Special cases

Now when ever the inner hypergeometric series in (2.1) to (2.3) and (3.1) to (3.3) can be summed up in terms of the Gamma product of the form

$$\frac{\Gamma(b - \mu s) \Gamma(1 - a + \mu s)}{\Gamma(1 - b + \mu s) \Gamma(a - \mu s)}.$$

We can evaluate the contour integral on the right hand side as either *H*-function or *G*-function and hence obtain the value of the integral involving the product of a polynomial and *H*-function or *G*-function in terms of a *H*-function or *G*-function respectively. We may take the following cases as illustrations:

(a) Taking $k=2, c=1+a+b;$

$$a_1 = \frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b; \quad b_1 = 1 + a$$

$$a_2 = 1 + \frac{1}{2}a + \frac{1}{2}b; \quad B = b + 1$$

then by virtue of (1.5) the generalized hypergeometric polynomial reduces to Jacobi polynomials, and summing the inner ${}_3F_2(+1)$ in the integrals of the right hand side by Saalschultz's theorem Bailey [1 p.9; 2.2(1)], the integral (2.1) reduces to.

$$(4.1) \quad \int_0^1 t^{\alpha-1} (1-t)^b P_n^{(a,b)}(1-2t) H_{p,q}^{u,v} \left[t^{l\mu} x \left| \begin{matrix} ((dp), (\mu p)) \\ ((eq), (\eta q)) \end{matrix} \right. \right] dt \\ = \frac{\Gamma(b+n+1)}{n!(l)^{1+b}} H_{p+2l, q+2l}^{u+l, v+l} \left[x \left| \begin{matrix} (\Delta(l, 1-\alpha)\mu), ((dp), (\mu p)), (\Delta(l, 1+a-\alpha)\mu) \\ (\Delta(l, 1+a-\alpha+n)\mu), ((eq), (\eta q)), (\Delta(l-b-\alpha-n)\mu) \end{matrix} \right. \right]$$

and integral (2.2) becomes

$$\int_0^{\pi/2} \sin^{2\alpha-1} \theta \cos^{2b+1} \theta P_n^{(a,b)}(\cos 2\theta) H_{p,q}^{u,v} \left[x \sin^{2l\mu} \theta \left| \begin{matrix} ((dp); (\mu p)) \\ ((eq); (\eta q)) \end{matrix} \right. \right] d\theta$$

$$\frac{1}{2} \frac{\Gamma(1+b+n)}{(l)^{1+b}} H_{p+2l, q+2l}^{u+l, v+l} \left[x \left| \begin{matrix} \{(\Delta(l, 1-\alpha)\mu); ((dp), (\mu p)), (\Delta(l, 1+a-\alpha), \mu)\} \\ \{(\Delta(l, 1+a-\alpha+n)\mu), ((eq), (\eta_q)), (\Delta(l-b-\alpha-n), \mu)\} \end{matrix} \right. \right]$$

and integral (3.1) reduces to Saxena [5 p.193(3.2)] with a slight change in representation.

$$(4.3) \quad \int_0^1 t^{\alpha-1} (1-t)^b P_n^{(a,b)}(1-2t) G_{p,q}^{u,v} \left(l^l x \left| \begin{matrix} (dp) \\ (eq) \end{matrix} \right. \right) dt \\ = \frac{\Gamma(1+b+n)}{n!(l)^{1+b}} G_{p+2l, q+2l}^{u+l, v+l} \left(x \left| \begin{matrix} \Delta(l, 1-\alpha), (dp), \Delta(l, 1+a-\alpha) \\ \Delta(l, 1+a-\alpha+n), (eq), \Delta(l, -b-\alpha-n) \end{matrix} \right. \right)$$

and integral (3.3) becomes

$$(4.4) \quad \int_0^{\pi/2} \sin^{2\alpha-1} \theta \cos^{2b+1} \theta P_n^{(a,b)}(\cos 2\theta) \times G_{p,q}^{u,v} \left(x \sin^{2l} \theta \left| \begin{matrix} (dp) \\ (eq) \end{matrix} \right. \right) d\theta \\ = \frac{1}{2} \frac{\Gamma(1+b+n)}{(l)^{1+b}} \times G_{p+2l, q+2l}^{u+l, v+l} \left(x \left| \begin{matrix} \Delta(l, 1-\alpha); (dp); \Delta(l+a-\alpha) \\ \Delta(l, 1+a-\alpha+n), (eq); \Delta(l, -b-\alpha-n) \end{matrix} \right. \right).$$

(b) By taking $k=1$, $a_1=\beta$, $a_2=c$; $b_1=\gamma$ and replacing α by p and β by $\beta-\gamma-n+1$ it can easily be seen that integrals (3.1) and (3.3) reduce to integrals (1) and (7) of Sharma [p.539, 541].

(c) Taking $k=1$, $c=1+a$, $p=0$, $q=0$ using (1.4) and summing the inner hypergeometric ${}_2F_1(+I)$ by Bailey [1 p.3] on the right hand side of (2.2) the integral (2.2) reduces to.

$$(4.5) \quad \int_0^\infty e^{-t} t^{\alpha-1} L_n^{(\alpha)}(x) H_{p,q}^{u,v} \left[t^{l\mu} x \left| \begin{matrix} ((dp); (\mu p)) \\ ((eq); (\eta_q)) \end{matrix} \right. \right] dt \\ = \frac{(2\pi)^{1/2(1-l)} (l)^{l+n-1/2}}{n!} H_{p+2l, q+2l}^{u+l, v+l} \left[t^{l\mu} x \left| \begin{matrix} \{(\Delta(l, 1-\alpha)\mu), ((dp), (\mu p)) \\ (\Delta(l, 1+a-2), \mu)\} \\ \{(\Delta(l, 1+a-\alpha+n), \mu), ((eq), (\eta_q))\} \end{matrix} \right. \right]$$

and (3.2) becomes

$$(4.6) \quad \int_0^\infty e^{-t} t^{\alpha-1} L_n^{(\alpha)}(x) G_{p,q}^{u,v} \left(t^l x \left| \begin{matrix} (dp) \\ (eq) \end{matrix} \right. \right) dt \\ = \frac{(2\pi)^{1/2(1-l)} (l)^{l+n-1/2}}{n!} G_{p+2l, q+2l}^{m+l, n+l} \left(t^l x \left| \begin{matrix} \Delta(l, 1-\alpha), (dp), \Delta(l, 1+a-\alpha) \\ \Delta(l, 1+a-\alpha+n), (eq) \end{matrix} \right. \right)$$

5. Similarly if we change the G -function to E -function and generalized hypergeometric function by virtue of Erdelyi [3 p.215(2)] and [3 p.208] respectively, all the above results can easily be reduced to the integral involving the generalized hypergeometric polynomial and E -function or generalized hypergeometric function.

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