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CONDUCTION OF HEAT IN A FINITE CIRCULAR CYLINDER

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1. Introduction

In heat engines of various kinds, the cylindrical solids have an important role to play and hence a study of temperature variation of these cylindrical solids which are used in the working of compound engines, air compressor, ordinary steam engines and internal combustion engines [4] are of great use.

Many authors have considered problems of heat conduction in which boundary conditions are prescribed (such as temperature or heat flux) and the temperature at internal points is required [1, 3, 4, 5, 6, 7, 8, 9] Masket and Vastano [3] have considered a problem of transient heat conduction in which they have obtained the boundary values from interior ones, and consequently they have named such problems as "Interior Value Problems (IVP)". Such problems arise in the aerodynamic heating and in indirect calorimetry devices for laboratory use. Recently Kalla [5] has considered a IVP problem of transient heat conduction in a finite circular cylinder with the given temperature distribution on any interior plane normal to the axis of the cylinder.

The object of the present paper is to consider a problem of heat conduction in a finite circular cylinder which is generating heat with the given temperature distribution on any interior plane normal to the axis of the cylinder, being a function of time and position and to determine the temperature at any point on one of the flat surfaces of the cylinder. The solution has been obtained with the help of Laplace and finite Hankel transforms [9]. The results obtained here are extension of the ones given recently by Kalla [5].

2. Integral transforms

We shall denote the classical Laplace transform of a function $f(r, z, t)$ as

$$\bar{f}(r, z, p)=\int_0^\infty e^{-pt} f(r, z, t)dt \tag{1}$$

The finite Hankel transform [9] of a function $f(r, z, t)$ will be denoted as

$$f_j(\alpha_i, z, t)=\int_0^a rf(r, z, t)J_0(r\alpha_i)dr \tag{2}$$

where J_0 is the Bessel function of the first kind of order zero and α_i is a root of the transcendental equation

$$J_0(a\alpha_i)=0 \tag{3}$$

If $f(r, z, t)$ satisfies Dirichlet's condition in the interval $(0, a)$ and if its finite Hankel transform in that range is defined as (2), then at any point of $(0, a)$ at which the function $f(r, z, t)$ is continuous [9]

$$f(r, z, t)=\frac{2}{a^2}\sum_i f_j(\alpha_i, z, t)\frac{J_0(r\alpha_i)}{[J_1(a\alpha_i)]^2} \tag{4}$$

where α_i is a root of the equation (3).

3. The solution

Let us consider the radial and axial heat flow in a circular cylinder bounded by the surfaces $z=0, z=h$ and $r=a$, and initially at temperature $n(r, s)$. Our problem may be described mathematically as to obtain the solution of the partial differential equation

$$\frac{1}{\kappa}\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\frac{\partial^2\theta}{\partial z^2}+A(r, z, t) \tag{5}$$

where κ is the diffusivity, subject to the following conditions

$$\begin{aligned} &\theta(r, 0, t)=m(r, t) && \text{to be found out}\\ &\theta(r, z, 0)=n(r, z) && t=0\\ &\theta(r, h, t)=p(r, t) && t>0\\ &\theta(a, z, t)=q(z, t) && t>0\\ &\theta(r, s, t) \quad \text{is known,} && 0<s<h \end{aligned} \tag{6}$$

Applying the finite Hankel transform to (5) and (6), we obtain

$$\frac{1}{\kappa}\frac{\partial\theta_j}{\partial t}=a\alpha_i q(z, t)J_i(a\alpha_i)-\alpha_i^2\theta_j+\frac{\partial^2\theta_j}{\partial z^2}+A_j(\alpha_i, z, t) \tag{7}$$

with the corresponding conditions

$$\begin{aligned} &\theta_j(\alpha_i, 0, t)=m_j(\alpha_i, t)\\ &\theta_j(\alpha_i, z, 0)=n_j(\alpha_i, z)\\ &\theta_j(\alpha_i, h, t)=p_j(\alpha_i, t) \end{aligned} \tag{8}$$

and $\theta_j(\alpha_i, s, t)$ which is known.

Now applying the Laplace transform (1) to equations (7) and (8) we get

$$\frac{d^2\bar{\theta}_j}{dz^2}-\left(\alpha_i^2+\frac{p}{\kappa}\right)\bar{\theta}_j+a\alpha_i\bar{q}(z,p)J_1(a\alpha_i)+\frac{1}{\kappa}\bar{n}_j(\alpha_i,p)+\bar{A}_j(\alpha_i,z,p)=0 \tag{9}$$

and

$$\begin{aligned}&\bar{\theta}_j(\alpha_i,0,p)=\bar{m}_j(\alpha_i,p)\\&\bar{\theta}_j(\alpha_i,h,p)=\bar{p}_j(\alpha_i,p)\\&\bar{\theta}_j(\alpha_i,s,p)\quad \text{which is known}\end{aligned} \tag{10}$$

From (9) using (10) we obtain

$$\begin{aligned}\bar{\theta}_j(\alpha_i,z,p)=&\,\bar{m}_j(\alpha_i,p)\frac{\sinh\left[(h-z)\left(\alpha_i^2+\frac{p}{\kappa}\right)^{\frac{1}{2}}\right]}{\sinh\left[h\left(\alpha_i^2+\frac{p}{\kappa}\right)^{\frac{1}{2}}\right]}\\&+\bar{p}_j(\alpha_i,p)\frac{\sinh\left[z\left(\alpha_i^2+\frac{p}{\kappa}\right)^{\frac{1}{2}}\right]}{\sinh\left[h\left(\alpha_i^2+\frac{p}{\kappa}\right)^{\frac{1}{2}}\right]}\\&-\frac{1}{f(D)}\left[a\alpha_iJ_i(a\alpha_i)\bar{q}(z,p)+\frac{1}{\kappa}\bar{n}_j(\alpha_i,z)+\bar{A}_j(\alpha_i,z,p)\right]\end{aligned} \tag{11}$$

where D is the usual differential operator $\frac{d}{dz}$ and $\frac{1}{f(D)}$ z is that function of z which when operated upon by $f(D)$ gives z.

After using the inverse Laplace and Hankel transforms, we obtain the solution of our problem,

$$\begin{aligned}m(r,t)=&\frac{4\pi\kappa}{a^2(h-s)^2}\sum_i\frac{J_0(\alpha_i,\ r)}{[J_i(\alpha_i,a)]^2}\sum_{m=1}^{\infty}(-1)^{m+1}m\\&\times\int_0^t\exp\left[-\frac{\{\alpha_i^2(h-s)^2+m^2\pi^2\}\kappa(t-T)}{(h-s)^2}\right]\left\{\sin\frac{m\pi h}{h-s}\theta_j(\alpha_i,\ s,T)\right.\\&\left.+\sin\frac{m\pi s}{h-s}p(\alpha_i,\ T)+\sin\frac{m\pi h}{h-s}\chi(s,T)\right\}\end{aligned} \tag{12}$$

where the summation is taken over all the positive roots of the equation (3), and

$$\bar{\chi}(s,p)=\frac{1}{f(D)}\left\{a\alpha_iJ_1(\alpha_i,\ a)\bar{q}(z,\ p)+\frac{1}{\kappa}n_j(\alpha_i,z)+\bar{A}_j(\alpha_i,z,p)\right\}$$

On the contrary, if $m(r,t)$ is known and the value of $\theta(r,z,t)$ $t>0$ is to be determined i.e. the temperature at any interval point is required, then the solution can be obtained as

$$\theta(r, z, t)=\frac{4\pi\kappa}{a^2h^2}\sum_i\frac{J_0(\alpha_i, r)}{[J_1(\alpha_i, a)]^2}\Big[\sum_{m=1}^{\infty}(-1)^m\sin\Big(\frac{m\pi z}{h}\Big)$$

$$\times\int_0^t\exp\Big\{-\frac{(\alpha_i^2h^2+m^2\pi^2)\kappa\,(t-T)}{h^2}\Big\}m_j(\alpha_i, T)dT$$

$$+\sin\Big(\frac{m\pi z}{h}\Big)\int_0^t\exp\Big\{-\frac{(\alpha_i^2h^2+m^2\pi^2)\kappa(t-T)}{h^2}\Big\}p_j(\alpha_i, T)dT\Big]$$

$$-\frac{2}{a^2}\sum_i\frac{J_0(\alpha_i, r)}{[J_1(\alpha_i, a)]^2}L^{-1}\{\chi(s, p)\}. \tag{13}$$

4. Particular cases

Let us consider some of the interesting particular cases of the general result (12). If we set $p(r, t)=q(z, t)=A(r, z, t)=0$ and that the initial distribution of temperature in the cylinder is μs, and $n(r, s)=\varepsilon s$, then

$$m(r, t)=\frac{4\pi\kappa}{a^2(h-s)^2}\sum_i\frac{J_0(\alpha_i, r)}{[J_1(\alpha_i, a)]^2}\frac{s}{\alpha_i}J_1(\alpha_i, a)$$

$$\times\sum_{m=1}^{\infty}(-1)^{m+1}m\sin\Big(\frac{m\pi h}{h-s}\Big)\Big[\mu a\frac{1-e^{-\gamma_i^2t}}{\gamma_i^2}-\varepsilon\frac{e^{-\alpha_i^2\kappa t}-e^{-\gamma_i^2t}}{\gamma_i^2-\alpha_i^2\kappa}\Big]$$

Similarly if we set $p(r, t)=q(z, t)=n(r, z)=0$ and $A(r, s, t)=\delta s$ then we obtain the temperature distribution as

$$m(r, t)=\frac{4\pi\kappa}{a^2(h-s)^2}\sum_i\frac{J_0(\alpha_i, r)}{[J_1(\alpha_i, a)]^2}\frac{as}{\alpha_i}J_1(\alpha_i, a)$$

$$\times\sum_{m=1}^{\infty}(-1)^{m+1}m\sin\Big(\frac{m\pi h}{h-s}\Big)\Big[\frac{1-e^{-\gamma_i^2t}}{\gamma_i^2}\Big(\mu-\frac{\delta}{\alpha_i^2}\Big)+\frac{\delta}{\alpha_i^2}\,\frac{e^{-\alpha_i^2\kappa t}-e^{-\gamma_i^2t}}{\gamma_i^2-\alpha_i^2\kappa}\Big]$$

where

$$\gamma_i^2=\kappa\Big[\alpha_i^2+\Big(\frac{m\pi}{h-s}\Big)^2\Big].$$

If the initial temperature in the cylinder is zero and there are no sources of heat present in the cylinder, then our solution (12) reduces to the one given earlier by Kalla [5].

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