Kyungpook Math. J.
Volume 16, Number 1
June, 1976

SOME TRANSFORMATION FORMULAE FOR HYPERGEOMETRIC SERIES

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1. Introduction

Carlitz [4], Abiodun & Sharma [7]; Singal [6] have given various transformation formulae for hypergeometric series of two variables. The object of this paper is to obtain four transformation formulae for hypergeometric series of two variables. Some interesting particular cases are also discussed.

The following notation due to Chaundy [8] will be used to represent the hypergeometric function of higher order and of two variables.

$$(1)\quad F\left[\begin{matrix}(a_p);\ (b_q);\ (c_r);x,\ y\\ (d_s);\ (e_h);\ (f_k)\ ;\end{matrix}\right]=\sum_{m,n=0}^{\infty}\frac{[(a_p)]_{m+n}[(b_q)]_m[(c_r)]_n x^m y^n}{[(d_s)]_{m+n}[(e_h)]_m[(f_k)]_n m!n!}$$

where (a_p) and $[(a_p)]_{m+n}$ will mean $a_1,\cdots,a_p$ and $(a_1)_{m+n},\cdots,(a_p)_{m+n}$.

2. The first transformation formula to be proved is

$$(2)\ F\left[\begin{matrix}-n;\ \gamma,\ \lambda;\ \delta,\ \mu;\ 2,\ 2\\ \beta;\quad 2\lambda;\quad 2\mu;\end{matrix}\right]$$

$$=\frac{(\beta-\gamma-\delta)_n}{(\beta)_n}F\left[\begin{matrix}-\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n;\ \frac{1}{2}\gamma,\ \frac{1}{2}+\frac{1}{2}\gamma;\ \frac{1}{2}\delta,\ \frac{1}{2}+\frac{1}{2}\delta;\ 1,\ 1\\ \frac{1}{2}(1-n-\beta+\gamma+\delta),\ \frac{1}{2}(2-n-\beta+\gamma+\delta);\ \lambda+\frac{1}{2};\ \mu+\frac{1}{2};\end{matrix}\right].$$

valid for $R(\beta-\gamma-\delta)>0$.

PROOF. To prove (2), we start with left side of (2).

$$F\left[\begin{matrix}-n;\ \gamma,\ \lambda;\ \delta,\ \mu;\ 2,2\\ \beta;\quad 2\lambda;\quad 2\mu;\end{matrix}\right]=\sum_{p=0}^{p+q}\sum_{q=0}^{\leq n}\frac{(-n)_{p+q}(\gamma)_p(\lambda)_p(\delta)_q(\mu)_q 2^{p+q}}{(\beta)_{p+q}(2\lambda)_p(2\mu)_q p!\ q!}$$

$$=\sum_{p=0}^{p+q}\sum_{q=0}^{\leq n}\frac{(-n)_{p+q}(\gamma)_p(\delta)_q}{(\beta)_{p+q}p!\ q!}\sum_{r=0}^{p/2}\frac{\left(-\frac{1}{2}p\right)_r\left(\frac{1}{2}-\frac{1}{2}p\right)_r}{\left(\lambda+\frac{1}{2}\right)_r r!}\sum_{s=0}^{q/2}\frac{\left(-\frac{1}{2}q\right)_s\left(\frac{1}{2}-\frac{1}{2}q\right)_s}{\left(\mu+\frac{1}{2}\right)_s s!}$$

we have used the formula, Rainville [1, p. 49]

$$(3)\quad {}_2F_1\left[\begin{matrix}-\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n;\ 1\\ b+\frac{1}{2}\ ;\end{matrix}\right]=\frac{2^n(b)_n}{(2b)_n}$$

$$=\sum_{p=0}^{p+q}\sum_{q=0}^{\leq n}\frac{(-n)_{p+q}(\gamma)_p(\delta)_q}{(\beta)_{p+q}\ p!\ \ q!}\sum_{r=0}^{p/2}\frac{(-p)_{2r}}{r!\left(\lambda+\frac{1}{2}\right)_r 2^{2r}}\sum_{s=0}^{q/2}\frac{(-q)_{2s}}{s!\left(\mu+\frac{1}{2}\right)_s 2^{2s}}$$

$$=\sum_{r=0}^{r+s}\sum_{s=0}^{\leq n}\frac{(-n)_{2r+2s}(\gamma)_{2r}(\delta)_{2s}}{(\beta)_{2r+2s}\left(\lambda+\frac{1}{2}\right)_r\left(\mu+\frac{1}{2}\right)_s r!\ s!\ 2^{2r+2s}}F_1[-n+2r+2s;\ \gamma+2r,\ \delta+2s;$$

$\beta+2r+2s;\ 1, 1]$

Now we use the formula due to Appell & Kampe de F'erit [5, p. 22, equation (24)]

$$(4)\qquad F_1[\alpha;\beta,\rho;\gamma;1,1]=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta-\rho)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta-\rho)},$$

valid for $R(\gamma-\alpha-\beta-\rho)>0$.

$$=\frac{(\beta-\gamma-\delta)_n}{(\beta)_n}F\left[\begin{matrix}-\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}n;\frac{1}{2}\gamma,\frac{1}{2}+\frac{1}{2}\gamma;\frac{1}{2}\delta,\frac{1}{2}+\frac{1}{2}\delta;\ 1,\ 1\\ \frac{1}{2}(1-n-\beta+\gamma+\delta),\frac{1}{2}(2-n-\beta+\gamma+\delta);\lambda+\frac{1}{2};\mu+\frac{1}{2};\end{matrix}\right].$$

This completes the proof of the formula.

3. The second transformation formula to be proved is

$$(5)\qquad F\left[\begin{matrix}\alpha;\ -m,\ \lambda;\ -n,\ \mu;\ 2,\ 2\\ \beta;\ \ 2\lambda;\ \ \mu;\end{matrix}\right]$$

$$=\frac{(\beta-\alpha)_{m+n}}{(\beta)_{m+n}}F\left[\begin{matrix}\frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha;\ -\frac{1}{2}m,\ \frac{1}{2}-\frac{1}{2}m;\ -\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}n;\ 1,\ 1\\ \frac{1}{2}(1+\alpha-\beta-m-n),\ \frac{1}{2}(2+\alpha-\beta-m-n);\ \lambda+\frac{1}{2};\mu+\frac{1}{2};\end{matrix}\right],$$

valid for $R(\beta-\alpha)>0$.

PROOF. The proof of (5) is same as that of (2). In case we put $\lambda=0$ in (5), we get a transforation formula between ${}_3F_2(2)$ and ${}_4F_3(1)$.

$$(6)\ {}_3F_2\left[\begin{matrix}\alpha,\ \mu,\ -n;\ 2\\ \beta,\ 2\mu;\end{matrix}\right]=$$

$$\frac{(\beta-\alpha)_n}{(\beta)_n}{}_4F_3\left[\begin{matrix}-\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n,\ \frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha;\ 1\\ \frac{1}{2}(1+\alpha-\beta-n),\ \frac{1}{2}(2+\alpha-\beta-n),\ \mu+\frac{1}{2};\end{matrix}\right].$$

with the help of (5), we write

$$(7)\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{m+n}}{m!\ n!}x^m y^n F\left[\begin{matrix}\alpha;\ -m,\ \lambda;\ -n,\ \mu;\ 2,\ 2\\ \beta;\ \ 2\lambda;\ \ 2\mu;\end{matrix}\right]$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta-\alpha)_{m+n}}{m!\ n!}x^m y^n$$

$$\times F\left[\begin{matrix}\frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha;\ -\frac{1}{2}m,\ \frac{1}{2}-\frac{1}{2}m;\ -\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n;\ 1,\ 1\\ \frac{1}{2}(1+\alpha-\beta-m-n),\ \frac{1}{2}(2+\alpha-\beta-m-n);\ \lambda+\frac{1}{2};\ \mu+\frac{1}{2};\end{matrix}\right].$$

First we simplify the left side of (7).

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha)_{r+s}(\lambda)_r(\mu)_s(-2x)^r(-2y)^s}{(2\lambda)_r(2\mu)_s\ r!\ s!}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta+r+s)_{m+n}}{m!\ n!}x^m y^n$$

$$=(1-x-y)^{-\beta}F_2\left[\alpha;\ \lambda,\ \mu;\ 2\lambda,\ 2\mu;\ \frac{2x}{x+y-1},\ \frac{2y}{x+y-1}\right].$$

Now we simplify the right side of (7)

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(\frac{1}{2}\alpha\right)_{r+s}\left(\frac{1}{2}+\frac{1}{2}\alpha\right)_{r+s}x^{2r}y^{2r}}{\left(\lambda+\frac{1}{2}\right)_r\left(\mu+\frac{1}{2}\right)_s r!\ s!}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta-\alpha)_{m+n}x^m y^n}{m!\ n!}$$

$$=(1-x-y)^{\alpha-\beta}F_4\left[\frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha;\ \lambda+\frac{1}{2},\ \mu+\frac{1}{2};\ x^2,\ y^2\right].$$

Thus we obtain

(8) $$F_4\left[\frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha;\ \lambda+\frac{1}{2},\ \mu+\frac{1}{2};x^2,\ y^2\right]$$
$$=(1-x-y)^{-\alpha}F_2\left[\alpha;\ \lambda,\ \mu;\ 2\lambda,\ 2\mu;\ \frac{2x}{x+y-1},\ \frac{2y}{x+y-1}\right].$$

(8) is due to Bailey [9].

4. The third transformation formula to be proved is

(9) $$F\left[\begin{matrix}\alpha;\ 1-m-\lambda,\ -m;\ 1-n-\mu,\ -n;\ -1,\ -1\\ \beta;\qquad \lambda\qquad ;\qquad \mu\qquad ;\end{matrix}\right]=\frac{(\beta-\alpha)_{m+n}}{(\beta)_{m+n}}$$

$$\times F\left[\begin{matrix}\alpha,\ 1-\beta-m-n;\ -\frac{1}{2}m,\ \frac{1}{2}-\frac{1}{2}m;\ -\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n;\ 1,\ 1\\ \frac{1}{2}(1-\beta+\alpha-m-n),\ \frac{1}{2}(2-\beta+\alpha-m-n);\ \lambda;\ \mu\end{matrix}\right].$$

valid for $R(\beta-\alpha)>0$.

PROOF. To prove (9), we start with the left side of (9).

$$F\left[\begin{matrix}\alpha;\ 1-m-\lambda,\ -m;\ 1-n-\mu,\ -n;\ -1,\ -1\\ \beta;\qquad \lambda\qquad ;\qquad \mu\qquad ;\end{matrix}\right]$$

$$=\sum_{p=0}^{m}\sum_{q=0}^{n}\frac{(-m)_p(1-m-\lambda)_p(-n)_q(1-n-\mu)_q(\alpha)_{p+q}(-1)^{p+q}}{(\beta)_{p+q}(\lambda)_p(\mu)_q\ p!\ q!}$$

$$=\sum_{p=0}^{m}\sum_{q=0}^{n}\frac{(\alpha)_{p+q}(-m)_p(-n)_q}{(\beta)_{p+q}\ p!\ q!}\sum_{r=0}^{m-p}\frac{(-p)_r(-m+p)_r}{(\lambda)_r\ r!}\sum_{s=0}^{n-q}\frac{(-q)_s(-n+q)_s}{(\mu)_s\ s!}$$

we have used the formua, Rainville [1, p. 69, ex. 5]

$$(10)\qquad {}_2F_1\left[\begin{matrix}-n, & a+n; & 1\\ & c & \end{matrix}\right]=\frac{(-1)^n(1+a-c)_n}{(c)_n}.$$

$$=\sum_{r=0}^{m/2}\sum_{s=0}^{n/2}\frac{(\alpha)_{r+s}\left(-\frac{1}{2}m\right)_r\left(\frac{1}{2}-\frac{1}{2}m\right)_r\left(-\frac{1}{2}n\right)_s\left(\frac{1}{2}-\frac{1}{2}n\right)_s 2^{2r+2s}(-1)^{r+s}}{(\beta)_{r+s}(\lambda)_r\ (\mu)_s\ r!\ s!}$$

$$\times F_1[\alpha+r+s;\ -m+2r,\ -n+2s;\ \beta+r+s;\ 1,\ 1]$$

Now we use (4).

$$=\frac{(\beta-\alpha)_{m+n}}{(\beta)_{m+n}}F\left[\begin{matrix}\alpha,\ 1-\beta-m-n;\ -\frac{1}{2}m,\ \frac{1}{2}-\frac{1}{2}m;\ -\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n;\ 1,\ 1\\ \frac{1}{2}(1-\beta+\alpha-m-n),\ \frac{1}{2}(2-\beta+\alpha-m-n);\ \lambda;\ \mu;\end{matrix}\right].$$

This completes the proof of the formulae. Taking $n=0$ in (9), it reduces to a known result due to Slater [2, p. 65. (2.4.21)].

5. The fourth transformation formula to be proved is

$$(11)\qquad F\left[\begin{matrix}\alpha;\ -m,\ \gamma;\ -n,\ \delta;\ 1,\ 1\\ \beta;\ 1-m-\gamma;\ 1-n-\delta;\end{matrix}\right]=\frac{(\beta-\alpha)_{m+n}}{(\beta)_{m+n}}$$

$$\times F\left[\begin{matrix}\alpha,\ 1-\beta-m-n;\ -\frac{1}{2}m,\ \frac{1}{2}-\frac{1}{2}m-\gamma;\ -\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n-\delta;\ 1,\ 1\\ \frac{1}{2}(1-\beta+\alpha-m-n),\ \frac{1}{2}(2-\beta+\alpha-m-n);\ 1-m-\gamma;\ 1-n-\delta;\end{matrix}\right],$$

valid for $R(\beta-\alpha)>0$.

PROOF. To prove (11), we start with the left side of (11).

$$F\left[\begin{matrix}\alpha;\ -m,\ \gamma;\ -n,\ \delta;\ 1,\ 1\\ \beta;\ 1-m-\gamma;\ 1-n-\delta;\end{matrix}\right]=\sum_{p=0}^{m}\sum_{q=0}^{n}\frac{(\alpha)_{p+q}(-m)_p(\gamma)_p(-n)_q(\delta)_q}{(\beta)_{p+q}(1-m-\gamma)_p(1-n-\delta)_q p!\ q!}$$

$$=\sum_{p=0}^{m}\sum_{q=0}^{n}\frac{(\alpha)_{p+q}(-m)_p(-n)_q}{(\beta)_{p+q}\ p!\ q!}\sum_{r=0}^{m-p}\frac{(-p)_r(-m+p)_r\left(\frac{1}{2}-\frac{1}{2}m-\gamma\right)_r}{(1-m-r)_r\left(\frac{1}{2}-\frac{1}{2}m\right)_r\ r!}$$

$$\times\sum_{s=0}^{n-q}\frac{(-q)_s(-n+q)_s\left(\frac{1}{2}-\frac{1}{2}n-\delta\right)_s}{(1-n-\delta)_s\left(\frac{1}{2}-\frac{1}{2}n\right)_s\ s!}$$

we have used the formula, Rainville [1, p. 87]

(12) $${}_3F_2\left[\begin{matrix}-n,\ a+n,\ \frac{1}{2}+\frac{1}{2}a-b;\ 1\\ 1+a-b,\ \frac{1}{2}+\frac{1}{2}a\qquad ;\end{matrix}\right]=\frac{(b)_n}{(1+a-b)_n},$$

where $1+a-b$ is a non-negative integer and a and b are independent of n.

$$=\sum_{r=0}^{m/2}\sum_{s=0}^{n/2}\frac{(\alpha)_{r+s}\left(\frac{1}{2}-\frac{1}{2}m-\gamma\right)_r\left(\frac{1}{2}-\frac{1}{2}n-\delta\right)_s\left(-\frac{1}{2}n\right)_s\left(-\frac{1}{2}m\right)_r}{(\beta)_{r+s}(1-m-\gamma)_r(1-n-\delta)_s}$$

$$\times\frac{(-1)^{r+s}2^{2r+2s}}{r!\ s!}F_1\ [\alpha+r+s;\ -m+2r,\ -n+2s;\ \beta+r+s;\ 1,\ 1].$$

Now we use (4).

$$=\frac{(\beta-\alpha)_{m+n}}{(\beta)_{m+n}}$$

$$\times F\left[\begin{matrix}\alpha,\ 1-\beta-m-n;\ -\frac{1}{2}m,\ \frac{1}{2}-\frac{1}{2}m-\gamma;\ -\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n-\delta;\ 1,\ 1\\ \frac{1}{2}(1-\beta+\alpha-m-n),\ \frac{1}{2}(2-\beta+\alpha-m-n);\ 1-m-\gamma;\ 1-m-\delta;\end{matrix}\right]$$

This completes the proof of the formula. If we put $m=0$ in (11), we have

(13) $${}_3F_2\left[\begin{matrix}-n,\ \alpha,\ \delta;\ 1\\ \beta,\ 1-n-\delta;\end{matrix}\right]$$

$$=\frac{(\beta-\alpha)_n}{(\beta)_n}{}_4F_3\left[\begin{matrix}-\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n-\delta,\ \alpha,\ 1-\beta-n;\ 1\\ \frac{1}{2}(1-\beta+\alpha-n),\ \frac{1}{2}(2-\beta+\alpha-n),\ 1-n-\delta;\end{matrix}\right].$$

Next we take $n=m$, $\gamma=\delta$, $\alpha=1-n-r$, $\beta=r+\delta$ and using the formula due to Carlitz [3]

(14) $$F\left[\begin{matrix}\alpha;\ \beta,\ -n;\ \gamma,\ -m;\ 1,\ 1\\ \beta+\gamma;\quad \alpha;\qquad \alpha;\end{matrix}\right]=\frac{(\alpha)_{m+n}(\beta)_n(\gamma)_m}{(\beta+\gamma)_{m+n}(\alpha)_n(\alpha)_n}$$

in (11), we get

(15) $$F\left[\begin{matrix}1-n-\gamma,\ 1-2n-\gamma;-\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}n-\gamma;\ -\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}n-\gamma;\ 1,\ 1\\ \frac{1}{2}(2-3\gamma-3n),\ \frac{1}{2}(3-3\gamma-3n);\ 1-n-\gamma;\ 1-n-\gamma;\end{matrix}\right]$$

$$=\frac{(-1)^n(\gamma)_n(1-\gamma)_n(3\gamma-1)_n}{(3\gamma-1)_{3n}}$$

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