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SOME NEW FORMULAE FOR HYPERGEOMETRIC SERIES

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1. Introduction

The object of this paper is to obtain some formulae for hypergeometric series of one variable. We use a special technique to derive these formulae from well-known results in special functions. The author has used this technique in the papers [2, 3] to obtain some new generating functions for Hermite, Gegenauer and Jacobi polynomials.

In the investigation we require the following result due to Sharma [2]

(1) ${}_pF_q[a_p;\, b_q;\, x]$

$$={}_{2p}F_{2q+1}\left[\frac{1}{2}a_p,\ \frac{1}{2}a_p+\frac{1}{2};\ \frac{1}{2},\ \frac{1}{2}b_q,\ \frac{1}{2}b_q+\frac{1}{2};\ (4)^{p-q-1}x^2\right]$$
$$+\frac{(a_p)}{(b_q)}x\ {}_{2p}F_{2q+1}\left[\frac{1}{2}a_p+\frac{1}{2},\ \frac{1}{2}a_p+1;\ \frac{3}{2},\ \frac{1}{2}b_q+\frac{1}{2},\ \frac{1}{2}b_q+1;(4)^{p-q-1}x^2\right],$$

where (a_p) stands for $a_1, \cdots, a_p$.

2. We consider the formula Erdelyi [1, p. 64, equ. (23)]

$$(2)\qquad (1-x)^{\alpha+\beta-\gamma}{}_2F_1\{\alpha,\beta;\gamma;\,x\}={}_2F_1\{\gamma-\alpha,\gamma-\beta;\,\gamma;\,x\}.$$

Multiply both sides of (2) by $(1-x)^{-\delta}$ and we rewrite (2) with the help of (1) as follows.

$$(3)\ \Big[{}_2F_1\Big\{\frac{1}{2}(\gamma+\delta-\alpha-\beta),\ \frac{1}{2}(\gamma+\delta-\alpha-\beta+1);\ \frac{1}{2};x^2\Big\}+(\gamma+\delta-\alpha-\beta)x$$
$$\times{}_2F_1\Big\{\frac{1}{2}(\gamma+\delta-\alpha-\beta+1),\ \frac{1}{2}(\gamma+\delta-\alpha-\beta+2);\ \frac{3}{2};\ x^2\Big\}\Big]\Big[{}_4F_3\Big\{\frac{1}{2}\alpha,$$
$$\frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\beta,\ \frac{1}{2}\beta+\frac{1}{2};\ \frac{1}{2},\ \frac{1}{2}\gamma,\ \frac{1}{2}\gamma+\frac{1}{2};x^2\Big\}+\frac{\alpha\beta}{\gamma}x\ {}_4F_3\Big\{\frac{1}{2}\alpha+\frac{1}{2},$$
$$\frac{1}{2}\alpha+1,\ \frac{1}{2}\beta+\frac{1}{2},\ \frac{1}{2}\beta+1;\ \frac{3}{2},\ \frac{1}{2}\gamma+\frac{1}{2},\ \frac{1}{2}\gamma+1;x^2\Big\}\Big]$$
$$=\Big[{}_2F_1\Big\{\frac{1}{2}\delta,\ \frac{1}{2}\delta+\frac{1}{2};\ \frac{1}{2};\ x^2\Big\}+\delta x\ {}_2F_1\Big\{\frac{1}{2}\delta+\frac{1}{2},\ \frac{1}{2}\delta+1;\ \frac{3}{2};x^2\Big\}\Big]$$
$$\times\Big[{}_4F_3\Big\{\frac{1}{2}(\gamma-\alpha),\ \frac{1}{2}(\gamma-\alpha+1),\ \frac{1}{2}(\gamma-\beta),\ \frac{1}{2}(\gamma-\beta+1);\ \frac{1}{2},\ \frac{1}{2}\gamma,$$

$$\tfrac{1}{2}\gamma+\tfrac{1}{2};x^2\Big\}+\frac{(\gamma-\alpha)\,(\gamma-\beta)}{\gamma}x\ {}_4F_3\Big\{\tfrac{1}{2}(\gamma-\alpha+1),\ \tfrac{1}{2}(\gamma-\alpha+2),$$
$$\tfrac{1}{2}(\gamma-\beta+1),\ \tfrac{1}{2}(\gamma-\beta+2);\ \tfrac{3}{2},\ \tfrac{1}{2}\gamma+\tfrac{1}{2},\ \tfrac{1}{2}\gamma+1;\ x^2\Big\}\Big].$$

Replacing x by ix in (3) and separating into real and imaginary parts, we have

$$(4)\ {}_2F_1\Big\{\gamma+\delta-\alpha-\beta,\ \gamma+\delta-\alpha-\beta+\tfrac{1}{2};\ \tfrac{1}{2};\ x\Big\}{}_4F_3\Big\{\alpha,\ \alpha+\tfrac{1}{2},\ \beta,\ \beta+\tfrac{1}{2};\ \tfrac{1}{2},\ \gamma,$$
$$\gamma+\tfrac{1}{2};\ x\Big\}+\frac{(\gamma+\delta-\alpha-\beta)}{\gamma}4\alpha\beta x\ {}_2F_1\Big\{\gamma+\delta-\alpha-\beta+\tfrac{1}{2},\ \gamma+\delta-\alpha-\beta+1;\ \tfrac{3}{2};\ x\Big\}$$
$$\times{}_4F_3\Big\{\alpha+\tfrac{1}{2},\ \alpha+1,\ \beta+\tfrac{1}{2},\beta+1;\ \tfrac{3}{2},\gamma+\tfrac{1}{2},\ \gamma+1;x\Big\}$$
$$={}_2F_1\Big\{\delta,\ \delta+\tfrac{1}{2},\ \tfrac{1}{2};\ x\Big\}\ {}_4F_3\Big\{\gamma-\alpha,\ \gamma-\alpha+\tfrac{1}{2},\ \gamma-\beta,\ \gamma-\beta+\tfrac{1}{2};\ \tfrac{1}{2},\ \gamma,$$
$$\gamma+\tfrac{1}{2};\ x\Big\}+\frac{4\delta(\gamma-\alpha)(\gamma-\beta)}{\gamma}x\ {}_2F_1\Big\{\delta+\tfrac{1}{2},\delta+1;\ \tfrac{3}{2};x\Big\}\ {}_4F_3\Big\{\gamma-\alpha+\tfrac{1}{2},$$
$$\gamma-\alpha+1,\ \gamma-\beta+\tfrac{1}{2},\ \gamma-\beta+1;\ \tfrac{3}{2},\ \gamma+\tfrac{1}{2},\ \gamma+1;\ x\Big\},\ \text{and}$$

$$(5)\ {}_2F_1\Big\{\gamma+\delta-\alpha-\beta+\tfrac{1}{2},\ \gamma+\delta-\alpha-\beta+1;\ \tfrac{3}{2};x\Big\}\ {}_4F_3\Big\{\alpha,\ \alpha+\tfrac{1}{2},\ \beta,\ \beta+\tfrac{1}{2};$$
$$\tfrac{1}{2},\ \gamma,\ \gamma+\tfrac{1}{2};\ x\Big\}2\{\gamma+\delta-\alpha-\beta\}+\frac{2\alpha\beta}{\gamma}{}_2F_1\Big\{\gamma+\delta-\alpha-\beta,\ \gamma+\delta-\alpha-\beta+\tfrac{1}{2};$$
$$\tfrac{1}{2};\ x\Big\}\ {}_4F_3\Big\{\alpha+\tfrac{1}{2},\ \alpha+1,\ \beta+\tfrac{1}{2},\ \beta+1;\ \tfrac{3}{2},\gamma+\tfrac{1}{2},\ \gamma+1;x\Big\}$$
$$=\frac{2(\gamma-\alpha)\,(\gamma-\beta)}{\gamma}\ {}_2F_1\Big\{\delta,\ \delta+\tfrac{1}{2}:\tfrac{1}{2};\ x\Big\}\ {}_4F_3\Big\{\gamma-\alpha+\tfrac{1}{2},\ \gamma-\alpha+1,\ \gamma-\beta+\tfrac{1}{2},$$
$$\gamma-\beta+1;\ \tfrac{3}{2},\ \gamma+\tfrac{1}{2},\ \gamma+1;x\Big\}+\delta\ {}_2F_1\Big\{\delta+\tfrac{1}{2},\ \delta+1;\ \tfrac{3}{2};\ x\Big\}{}_4F_3\Big\{\gamma-\alpha,\ \gamma-\alpha$$
$$+\tfrac{1}{2},\ \gamma-\beta,\ \gamma-\beta+\tfrac{1}{2}:\tfrac{1}{2},\ \gamma,\ \gamma+\tfrac{1}{2};\ x\Big\}.$$

In particular, we put $\delta=0$ in (4) and (5), we get the following interesting formulae.

$$(6)\ {}_4F_3\Big\{\gamma-\alpha,\ \gamma-\alpha+\tfrac{1}{2},\ \gamma-\beta,\ \gamma-\beta+\tfrac{1}{2}:\tfrac{1}{2},\ \gamma,\ \gamma+\tfrac{1}{2};\ x\Big\}$$
$$={}_2F_1\Big\{\gamma-\alpha-\beta,\ \gamma-\alpha-\beta+\tfrac{1}{2}:\tfrac{1}{2};\ x\Big\}\ {}_4F_3\Big\{\alpha,\ \alpha+\tfrac{1}{2},\ \beta,\ \beta+\tfrac{1}{2}:\tfrac{1}{2},\ \gamma,$$
$$\gamma+\tfrac{1}{2};\ x\Big\}+\frac{4\alpha\beta(\gamma-\alpha-\beta)}{\gamma}x\ {}_2F_1\Big\{\gamma-\alpha-\beta+\tfrac{1}{2},\ \gamma-\alpha-\beta+1;\ \tfrac{3}{2};\ x\Big\}{}_4F_3\Big\{\alpha+\tfrac{1}{2},$$
$$\alpha+1,\ \beta+\tfrac{1}{2},\ \beta+1;\tfrac{3}{2},\ \gamma+\tfrac{1}{2},\ \gamma+1;x\Big\},\ \text{and}$$

$$(7)\ {}_4F_3\Big\{\gamma-\alpha+\tfrac{1}{2},\ \gamma-\alpha+1,\ \gamma-\beta+\tfrac{1}{2},\ \gamma-\beta+1;\ \tfrac{3}{2},\ \gamma+\tfrac{1}{2},\ \gamma+1;x\Big\}$$

$$=\frac{\gamma(\gamma-\alpha-\beta)}{(\gamma-\alpha)(\gamma-\beta)}\ {}_2F_1\left\{\gamma-\alpha-\beta+\frac{1}{2},\ \gamma-\alpha-\beta+1;\ \frac{3}{2};\ x\right\}{}_4F_3\left\{\alpha,\ \alpha+\frac{1}{2},\ \beta,\right.$$
$$\left.\beta+\frac{1}{2}:\frac{1}{2},\ \gamma,\ \gamma+\frac{1}{2};\ x\right\}+\frac{\alpha\beta}{(\gamma-\alpha)(\gamma-\beta)2}F_1\left\{\gamma-\alpha-\beta,\ \gamma-\alpha-\beta+\frac{1}{2}:\frac{1}{2};\ x\right\}$$
$$\times{}_4F_3\left\{\alpha+\frac{1}{2},\ \alpha+1,\ \beta+\frac{1}{2},\ \beta+1;\ \frac{3}{2},\ \gamma+\frac{1}{2},\ \gamma+1;x\right\}.$$

Next we consider Kummers first formula, Rainville [4, p. 125, equ(2)].

(7) $\quad {}_1F_1\{\alpha;\beta;ax\}=e^{ax}\ {}_1F_1\{\beta-\alpha;\beta;-ax\}.$

Multiplying both sides by e^{bx} and rewriting (4) with the help of (1), we have

(8) $$\left[{}_0F_1\left\{-;\ \frac{1}{2}:\frac{1}{4}b^2x^2\right\}+bx\ {}_0F_1\left\{-;\ \frac{3}{2}:\frac{1}{4}b^2x^2\right\}\right]\left[{}_2F_3\left\{\frac{1}{2}\alpha,\ \frac{1}{2}\alpha+\frac{1}{2}:\frac{1}{2},\right.\right.$$
$$\left.\frac{1}{2}\beta,\ \frac{1}{2}\beta+\frac{1}{2}:\frac{1}{4}a^2x^2\right\}+\frac{\alpha ax}{\beta}{}_2F_3\left\{\frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\alpha+1;\ \frac{3}{2},\ \frac{1}{2}\beta+\frac{1}{2},\right.$$
$$\left.\left.\frac{1}{2}\beta+1;\ \frac{1}{4}a^2x^2\right\}\right]$$
$$=\left[{}_0F_1\left\{-;\ \frac{1}{2}:\frac{1}{4}x^2(a+b)^2\right\}+(a+b)x\ {}_0F_1\left\{-;\frac{3}{2}:\frac{1}{4}(a+b)^2x^2\right\}\right]\ \left[{}_2F_3\left\{\frac{1}{2}(\beta\right.\right.$$
$$\left.-\alpha),\ \frac{1}{2}(\beta-\alpha+1);\ \frac{1}{2},\ \frac{1}{2}\beta,\ \frac{1}{2}\beta+\frac{1}{2}:\frac{1}{4}a^2x^2\right\}+\frac{(\beta-\alpha)ax}{\beta}{}_2F_3\left\{\frac{1}{2}(\beta\right.$$
$$\left.\left.-\alpha+1),\ \frac{1}{2}(\beta-\alpha+2);\ \frac{3}{2},\ \frac{1}{2}\beta+\frac{1}{2},\ \frac{1}{2}\beta+1;\ \frac{1}{4}a^2x^2\right\}\right].$$

Replacing x by ix and comparing the real and imaginary parts of (δ), we have

(9) $${}_0F_1\left\{-;\frac{1}{2}:\frac{1}{4}b^2\right\}\ {}_2F_3\left\{\alpha,\ \alpha+\frac{1}{2}:\frac{1}{2},\ \beta,\ \beta+\frac{1}{2}:\frac{1}{4}a^2\right\}+\frac{\alpha ab}{\beta}\ {}_0F_1\left\{-;\right.$$
$$\left.\frac{3}{2}:\frac{1}{4}b^2\right\}\ {}_2F_3\left\{\alpha+\frac{1}{2},\ \alpha+1;\ \frac{3}{2},\ \beta+\frac{1}{2},\ \beta+1;\ \frac{1}{4}a^2\right\}$$
$$={}_0F_1\left\{-;\frac{1}{2}:\frac{1}{4}(a+b)^2\right\}\ {}_2F_3\left\{\beta-\alpha,\beta-\alpha+\frac{1}{2}:\frac{1}{2},\ \beta,\beta+\frac{1}{2}:\frac{1}{4}a^2\right\}$$
$$+\frac{a(\beta-\alpha)(a+b)}{\beta}{}_0F_1\left\{-;\ \frac{3}{2}:\frac{1}{4}(a+b)^2\right\}\ {}_2F_3\left\{\beta-\alpha+\frac{1}{2},\ \ \beta-\alpha+1;\ \frac{3}{2},\right.$$
$$\left.\beta+\frac{1}{2},\ \beta+1;\ \frac{1}{4}a^2\right\},\ \text{and}$$

(10) $$b\ {}_0F_1\left\{-;\frac{3}{2};\frac{1}{4}b^2\right\}\ {}_2F_3\left\{\alpha,\ \alpha+\frac{1}{2};\frac{1}{2},\ \beta,\ \beta+\frac{1}{2};\frac{1}{4}a^2\right\}$$
$$+\frac{\alpha a}{\beta}\ {}_0F_1\left\{-;\frac{1}{2}:\frac{1}{4}b^2\right\}\ {}_2F_3\left\{\alpha+\frac{1}{2},\ \alpha+1;\ \frac{3}{2},\ \beta+\frac{1}{2},\ \beta+1;\frac{1}{4}a^2\right\}$$
$$=(a+b)\ {}_0F_1\left\{-;\frac{3}{2}:\frac{1}{4}(a+b)^2\right\}\ {}_2F_3\left\{\beta-\alpha,\ \beta-\alpha+\frac{1}{2}:\frac{1}{2},\ \beta,\ \beta+\frac{1}{2};\right.$$

$\frac{1}{4}a^2\Big\} + \frac{(\beta-\alpha)a}{\beta}\ {}_0F_1\Big\{-\ ;\ \frac{1}{2}\ ;\ \frac{1}{4}(a+b)^2\Big\}\ {}_2F_3\Big\{\beta-\alpha+\frac{1}{2},\ \beta-\alpha+1;$
$\frac{3}{2},\ \beta+\frac{1}{2},\ \beta+1;\ \frac{1}{4}a^2\Big\}.$

We shall mention below some particular cases of (9) and (10). Taking $b=0$ in (9) and (10), we get

$$(11)\ {}_2F_3\Big\{\alpha,\ \alpha+\frac{1}{2}\ ;\ \frac{1}{2},\ \beta,\ \beta+\frac{1}{2};\ x\Big\} = {}_0F_1\Big\{-\ ;\ \frac{1}{2};\ x\Big\}\ {}_2F_3\Big\{\beta-\alpha,\ \beta-\alpha+\frac{1}{2}\ ;\ \frac{1}{2},\ \beta,\ \beta+\frac{1}{2}\ ;\ x\Big\} + \frac{4x(\beta-\alpha)}{\beta}\ {}_0F_1\Big\{-\ ;\ \frac{3}{2}\ ;\ x\Big\}{}_2F_3\Big\{\beta-\alpha+\frac{1}{2},\ \beta-\alpha+1;\ \frac{3}{2},\ \beta+\frac{1}{2},\ \beta+1;\ x\Big\},$$ and

$$(12)\ {}_2F_3\Big\{\alpha+\frac{1}{2},\ \alpha+1;\ \frac{3}{2},\ \beta+\frac{1}{2},\ \beta+1;\ x\Big\} = \frac{\beta}{\alpha}\ {}_0F_1\Big\{-\ ;\ \frac{3}{2}\ ;\ x\Big\}{}_2F_3\Big\{\beta-\alpha,\ \beta-\alpha+\frac{1}{2}\ ;\ \frac{1}{2},\ \beta,\ \beta+\frac{1}{2}\ ;\ x\Big\} + \frac{\beta-\alpha}{\alpha}\ {}_0F_1\Big\{-\ ;\ \frac{1}{2};\ x\Big\}{}_2F_3\Big\{\beta-\alpha+\frac{1}{2},\ \beta-\alpha+1;\ \frac{3}{2},\ \beta+\frac{1}{2},\ \beta+1;\ x\Big\}.$$

If we take $\beta=\alpha$ in (9) and (10), then we have

$$(13)\ {}_0F_1\Big\{-\ ;\ \frac{1}{2}\ ;\ (x+y)^2\Big\} = {}_0F_1\Big\{-\ ;\ \frac{1}{2}\ ;\ x^2\Big\}{}_0F_1\Big\{-\ ;\ \frac{1}{2}\ ;\ y^2\Big\} + 4xy\ {}_0F_1\Big\{-;\frac{3}{2};x^2\Big\}\ {}_0F_1\Big\{-;\frac{3}{2};y^2\Big\},$$ and

$$(14)\ (x+y)\ {}_0F_1\Big\{-;\frac{3}{2}\ ;\ (x+y)^2\Big\} = x\ {}_0F_1\Big\{-;\frac{3}{2};\ x^2\Big\}{}_0F_1\Big\{-;\ \frac{1}{2};y^2\Big\} + y\ {}_0F_1\Big\{-;\frac{1}{2};x^2\Big\}\ {}_0F_1\Big\{-;\frac{3}{2};y^2\Big\}.$$

We can easily obtain the generalization of (13) and (14) in the following forms.

$$(15)\ {}_2F_1\Big\{\lambda,\ \lambda+\frac{1}{2}\ ;\ \frac{1}{2}\ ;\ (x+y)^2\Big\} = F_4\Big\{\lambda,\ \lambda+\frac{1}{2}\ ;\ \frac{1}{2},\ \frac{1}{2};x^2,y^2\Big\} + (2\lambda+1)2\lambda xyF_4\Big\{\lambda+1,\ \lambda+\frac{3}{2}\ ;\ \frac{3}{2},\ \frac{3}{2};x^2,y^2\Big\},$$ and

$$(16)\ (x+y){}_2F_1\Big\{\lambda+\frac{1}{2},\ \lambda;\ \frac{3}{2}\ ;\ (x+y)^2\Big\} = xF_4\Big\{\lambda+\frac{1}{2},\ \lambda;\ \frac{3}{2},\ \frac{1}{2};x^2,y^2\Big\} + yF_4\Big\{\lambda+\frac{1}{2},\ \lambda;\ \frac{1}{2},\ \frac{3}{2};x^2,y^2\Big\}.$$

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