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ON SOME METHODS OF SUMMABILITY

By Ashok Kumar

1. Introduction

The concept of generalised Nörlund summability (N,p,α) has been introduced and investigated by Borwein [1] and Das [3,4]. In [4], Das has considered the most general problem of relative effectiveness of (N,p,α) and (N,q,β) methods and has established various inclusion and equivalence theorems.

In the present paper, we further investigate the some problem and prove some more inclusion and equivalence theorems. It is interesting to note that our results generalise many known results including those of Borwein and Cass [2, Theorems 3 and 5], and Cesaro and Hardy [5, Theorems 14 and 23]. Incidently, some new results of inclusion and equivalence are obtained as particular cases of our theorems.

2. Preliminaries

We define the convolution $(p*\alpha)_n$ of two sequences $\{p_n\}$ and $\{\alpha_n\}$ of real numbers as

$$(p*\alpha)_n=\sum_{\nu=0}^{n} p_{n-\nu}\alpha_\nu.$$

We shall make use of the fact that the operation of convolution is commutative and associative.

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with $\{s_n\}$ as the sequence of partial sums. Let $\{p_n\}$ and $\{\alpha_n\}$ be sequences of real numbers such that

$$(p*\alpha)_n\neq 0 \text{ for } n\geq 0, \quad =0 \text{ for } n<0.$$

Then $\sum_{n=0}^{\infty} a_n$(or $\{s_n\}$) is said to be summable by the *generalised Nörlund method* (N,p,α) to the value s, if

$$\frac{(p*\alpha s)_n}{(p*\alpha)_n}\to s \quad \text{as} \quad n\to\infty,$$

and is denoted by

$$\sum_{n=0}^{\infty} a_n = s \ (N, p, \alpha) \text{ or } s_n \to s \ (N, p, \alpha).$$

See for example [4].

The method (N, p, α) reduces to the Nörlund method (N, p) when $\alpha_n = 1$ [5, p. 64] and to the method (N, α) when $p_n = 1$ [5, p. 57].

The method (N, p, α) is said to be *regular* if it sums every convergent series to its ordinary sum. If $p_n > 0$, $\alpha_n > 0$ for $n \geq 0$, then the necessary and sufficient condition for the regularity of (N, p, α) method is

$$p_{n-\nu} = o((p*\alpha)_n)$$

as $n \to \infty$ (ν fixed), (see [4]).

Given any sequence $\{p_n\}$, we write

$$p(z) = \sum_{n=0}^{\infty} p_n z^n$$

whenever the series on the right converges. We define the sequence $\{k_n\}$ of constants by means of the formal identity

$$k(z) = \frac{q(z)}{p(z)}, \quad k_{-1} = 0.$$

As usual we say that the sequence $\{p_n\} \in \mathfrak{M}$, if

$$p_0 = 1, \ p_n > 0, \ \frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1 \text{ for } n = 0, 1, 2, \cdots\cdots.$$

If P and Q are methods of summability, we say that Q is more *effective* than P if every series summable P is also summable Q to the same sum and write $P \Longrightarrow Q$. If $P \Longrightarrow Q$ and $Q \Longrightarrow P$, then we say that the methods are *equivalent* and write $P \Longleftrightarrow Q$.

In the rest of the paper it is assumed that $p_n > 0$, $q_n > 0$, $\alpha_n > 0$, $\beta_n > 0$ for $n \geq 0$.

3. The Lemmas

In order to prove our theorems we need a few lemmas.

LEMMA 1. *Suppose that* $\{p_n\} \in \mathfrak{M}$.

(a) *If*

$$\frac{p_n}{p_{n-1}} \leq \frac{q_n}{q_{n-1}} \quad \textit{for } n > 0, \tag{1}$$

then $k_0 > 0$ *and* $k_n \geq 0$ *for* $n > 0$.

(b) *If*

$$\frac{q_n}{q_{n-1}} \leq \frac{p_n}{p_{n-1}} \quad for\ n>0, \tag{2}$$

then $k_0>0$ *and* $k_n \leq 0$ *for* $n>0$.

The proofs of (a) and (b) are respectively contained in the proofs of Theorem 23 of [5] and Theorem 3 of [2].

LEMMA 2. *Let* $\{p_n\} \in \mathfrak{M}$, $q_n=O(p_n)$ *and* (1) *hold. Then, if* (N, q, α) *is regular,* (N, p, α) *is regular.*

By virtue of (1), it can be easily verified that

$$q_n=O(p_n) \text{ implies } (q*\alpha)_n=O((p*\alpha)_n). \tag{3}$$

Since, by definition, $q_n=(k*p)_n$, therefore, by Lemma 1 (a), $k_0 p_n \leq q_n$. Thus, using (3) and the regularity of (N, q, α), we find that, for ν fixed,

$$p_{n-\nu}=o((p*\alpha)_n),$$

which proves that (N, p, α) is regular.

LEMMA 3. *The inclusion* $(N, p, \alpha) \Longrightarrow (N, q, \alpha)$ *holds if and only if*

$$(|k|*|p*\alpha|)_n=O((q*\alpha)_n)$$

and

$$k_{n-\nu}=o((q*\alpha)_n)$$

as $n\to\infty$ *(*ν *fixed).*

This is Lemma 1 of [4] with $\alpha_n=\beta_n$ for every n.

In proving our theorems, we shall very frequently appeal to Theorem 1 of [4], and so for the sake of completeness we state it here as:

LEMMA 4. *Let* $\{p_n\} \in \mathfrak{M}$, (1) *hold and let either the set of conditions*

$$A_1 : \frac{\beta_n}{\alpha_n} \geq \frac{\beta_{n+1}}{\alpha_{n+1}},$$

$$A_2 : (N, q, \beta) \text{ is regular,}$$

or the set of conditions

$$B_1 : \frac{\beta_n}{\alpha_n} \leq \frac{\beta_{n+1}}{\alpha_{n+1}},$$

$$\mathrm{B}_2 : \frac{\beta_n (q*\alpha)_n}{\alpha_n (q*\beta)_n} = O(1),$$

$$\mathrm{B}_3 : (N, q, \alpha) \text{ is regular,}$$

hold. Then

$$(N, p, \alpha) \Longrightarrow (N, q, \beta).$$

4. Main results

THEOREM 1. *If* $\{p_n\} \in \mathfrak{M}$, $p_n = O(q_n)$, (N, q, α) *is regular and*

$$\frac{q_n}{q_{n-1}} \leq \frac{p_n}{p_{n-1}} \text{ for } n > n_0, \tag{4}$$

then (N, p, α) *is regular and*

$$(N, p, \alpha) \Longrightarrow (N, q, \alpha).$$

PROOF. Case $n_0 = 0$. Using Lemma 1(b), we have

$$(q*\alpha)_n \leq k_0 (p*\alpha)_n,$$

and since

$$p_n \leq H q_n$$

(where H is a positive constant), therefore, for fixed ν,

$$\frac{p_{n-\nu}}{(p*\alpha)_n} \leq k_0 H \frac{q_{n-\nu}}{(q*\alpha)_n}$$
$$= o(1)$$

by the regularity of (N, q, α). Thus (N, p, α) is regular.

Now

$$(|k|*p)_n = k_0 p_n - k_1 p_{n-1} - \cdots - k_n p_0 = 2k_0 p_n - q_n \leq 2k_0 p_n + q_n = O(q_n) \tag{5}$$

since $p_n = O(q_n)$. Using (5) and the regularity of (N, q, α), we obtain

$$|k_n| p_0 \leq (|k|*p)_n = O(q_n) = o((q*\alpha)_n).$$

Since $0 < (q*\alpha)_{n-\nu} \leq (q*\alpha)_n$ for $\nu > 0$, so

$$k_{n-\nu} = o((q*\alpha)_n).$$

Further

$$(|k|*p*\alpha)_n = 2k_0 (p*\alpha)_n - (k*p*\alpha)_n \leq 2k_0 (p*\alpha)_n + (q*\alpha)_n = O((q*\alpha)_n)$$

since, by using (4), it can easily be verified that

$$p_n = O(q_n) \text{ implies } (p*\alpha)_n = O((q*\alpha)_n).$$

The result now follows by Lemma 3.

For the general case, we construct a sequence $\{r_n\}$ in the following manner. We have

$$\frac{q_n}{q_{n-1}} \leq \frac{p_n}{p_{n-1}} \quad \text{for } n=n_0+1, n_0+2, \cdots$$

Write

$$t_n = q_n \quad \text{for } n=n_0, \ n_0+1, \cdots.$$

and define t_n recursively for $n=n_0-1$, $n_0-2, \cdots, 0$, such that $t_n>0$, and

$$\frac{t_{n+1}}{t_n} \leq \min\left(\frac{t_{n+2}}{t_{n+1}}, \ \frac{q_{n+1}}{q_n}, \ \frac{p_{n+1}}{p_n}\right).$$

Now setting $r_n = \frac{t_n}{t_0}$, we find that

$$\{r_n\} \in \mathfrak{M}, \ \frac{r_n}{r_{n-1}} \leq \frac{q_n}{q_{n-1}}, \ \frac{r_n}{r_{n-1}} \leq \frac{p_n}{p_{n-1}} \ \text{for } n>0.$$

We also have $q_n = O(r_n)$.

Now, since (N, r, α) is regular (by Lemma 2), and $p_n = O(q_n) = O(r_n)$, therefore, by the case $n_0=0$, it follows that (N, p, α) is regular and

$$(N, p, \alpha) \Longrightarrow (N, r, \alpha). \tag{6}$$

Further, by Lemma 4 with $\alpha_n = \beta_n$ and p_n replaced by r_n, we obtain

$$(N, r, \alpha) \Longrightarrow (N, q, \alpha). \tag{7}$$

The result follows from (6) and (7).

THEOREM 2. *If, in addition to the hypotheses of Theorem* 1, $\{q_n\} \in \mathfrak{M}$, *then*

$$(N, p, \alpha) \Longleftrightarrow (N, q, \alpha).$$

PROOF. It has been established in Theorem 1 that, under the given hypotheses, (N, p, α) is regular.

Now, in the case $n_0=0$, taking $\alpha_n = \beta_n$ and interchanging p_n and q_n in Lemma 4 we obtain $(N, q, \alpha) \Longrightarrow (N, p, \alpha)$. This in conjunction with the case $n_0=0$ of Theorem 1 yields the result.

For the general case define $\{r_n\}$ as in the proof of Theorem 1. Interchanging p_n and q_n, and then writing r_n for q_n in Lemma 4 (with $\alpha_n = \beta_n$), we obtain $(N, r, \alpha) \Longrightarrow (N, p, \alpha)$.

Further, since (N, r, α) is regular (by Lemma 2) and $q_n = O(r_n)$, so by the case

$n_0=0$ of Theorem 1, $(N, q, \alpha) \Longrightarrow (N, r, \alpha)$ (this is obtained by interchanging p_n and q_n, and then writing r_n for p_n in Theorem 1). Thus

$$(N, q, \alpha) \Longrightarrow (N, p, \alpha).$$

Combining this with Theorem 1, we obtain the desired result.

THEOREM 3. *Suppose that* $\{p_n\}\in\mathfrak{M}$, $\{q_n\}\in\mathfrak{M}$, $p_n=O(q_n)$ *and that* (4) *holds. Suppose also that either the set of conditions* A *or the set of conditions* B *of Lemma* 4 *hold. Then*

$$(N, h, \alpha) \Longrightarrow (N, q, \beta).$$

PROOF. We claim that under the given sets of conditions A and B, (N, q, α) is regular. This is included in the set of conditions B. Under the set of conditions A, since

$$(q*\alpha)_n \geq \frac{\alpha_0}{\beta_0}(q*\beta)_n \qquad \text{(by } A_1\text{)},$$

therefore, for fixed ν,

$$\frac{q_{n-\nu}}{(q*\alpha)_n} \leq \frac{\beta_0}{\alpha_0}\frac{q_{n-\nu}}{(q*\beta)_n}=o(1) \qquad \text{(by } A_2\text{)},$$

which implies that (N, q, α) is regular.

Now, in the case $n_0=0$, $(N, p, \alpha) \Longrightarrow (N, q, \alpha)$ by Theorem 1 and $(N, q, \alpha) \Longrightarrow (N, q, \beta)$ by Lemma 4 (with $p_n=q_n$), and thus

$$(N, p, \alpha) \Longrightarrow (N, q, \beta).$$

For the general case, define $\{r_n\}$ as in the proof of Theorem 1. Again, by Theorem 1, $(N, p, \alpha) \Longrightarrow (N, r, \alpha)$, and by Lemma 4, $(N, r, \alpha) \Longrightarrow (N, q, \beta)$. Hence $(N, p, \alpha) \Longrightarrow (N, q, \beta)$. This completes the proof.

THEOREM 4. *Let* $\{p_n\}\in\mathfrak{M}$ *and let*

$$\frac{p_n}{p_{n-1}} \leq \frac{q_n}{q_{n-1}} \text{ for } n>n_0.$$

Suppose that either the set of conditions A *or the set of conditions* B *of Lemma* 4 *hold.*

If (N, p, α) *is regular, then*

$$(N, p, \alpha) \Longrightarrow (N, q, \beta).$$

PROOF. The case $n_0=0$ is Theorem 1 in [4]. It is to be noted that, in this case, we do not require the regularity of (N, p, α) method.

For the general case, interchanging p_n and q_n in the construction of $\{r_n\}$ (cf. the proof of Theorem 1), we obtain

$$\{r_n\}\in\mathfrak{M}, \frac{r_n}{r_{n-1}}\leq\frac{p_n}{p_{n-1}}, \quad \frac{r_n}{r_{n-1}}\leq\frac{q_n}{q_{n-1}} \quad \text{for } n>0$$

and

$$p_n=O(r_n). \tag{8}$$

Because of (8) and the fact that (N, p, α) is regular, it follows from Lemma 2 that (N, r, α) is regular. Thus, by the case $n_0=0$ of Theorem 1, $(N, p, \alpha) \Longrightarrow (N, r, \alpha)$; and by Lemma 4 (with p_n replaced by r_n) $(N, r, \alpha)\Longrightarrow(N, q, \beta)$. Hence

$$(N, p, \alpha) \Longrightarrow (N, q, \beta)$$

as required

COROLLARY 1. *Suppose that the hypotheses of Theorem 4 are satisfied. Then, if* $q_n=O(p_n)$,

$$(N, p, \alpha) \Longrightarrow (N, q, \beta).$$

PROOF. Since $q_n=O(p_n)=O(r_n)$(by(8)) and since (N, q, α) is regular (cf. the proof of Theorem 3), it follows from Lemma 2 that (N, r, α) is regular, and hence (N, p, α) is regular (by Theorem 1). The result now follows from Theorem 4.

5. Special cases

As particular instances of our theorems, we obtain following known and unknown results.

Taking $\alpha_n=1$ in Theorem 1, we obtain Theorem 3 of [2]. It is worth mentioning that in [2] the only case in which (4) holds for $n_0=0$ has been considered. Also putting $p_n=1$, we obtain

THEOREM 1′. *If* (N, q, α) *is regular,* $\frac{1}{q_n}=O(1)$ *and* $q_n\leq q_{n-1}$ *for* $n>n_0$; *then*

$$(\overline{N}, \alpha) \Longrightarrow (N, q, \alpha).$$

The case $n_0=0$ of Theorem 1′ is [4, Theorem 3(ii)]; for, in this case,

$$1=O(q_n) \text{ implies } (1*\alpha)_n=O((q*\alpha)_n).$$

By taking $\alpha_n=1$ and interchanging p_n and q_n in Theorem 2 we deduce first clause of Theorem 5 in [2]. Another important special case of Theorem 2 is

THEOREM 2′. *If* $\{p_n\}\in\mathfrak{M}$, $\frac{1}{p_n}=O(1)$ *and* (N,p,α) *is regular, then*

$$(\overline{N},\alpha)\Longleftrightarrow(N,p,\alpha).$$

This is obtained by putting $p_n=1$ and then writing p_n for q_n. Under slightly different conditions, a part of the result of Theorem 2′, namely $(N,p,\alpha)\Longrightarrow(N,\alpha)$, has been established by Das [3, Theorem 4].

By substituting $p_n=1, \beta_n=1$ in Theorem 3, we obtain

THEOREM 3′. *Let* $\{q_n\}\in\mathfrak{M}$ *and* $\frac{1}{q_n}=O(1)$. *Suppose that either*

$$\mathrm{A}:\alpha_{n+1}\geq\alpha_n;$$

or

$$\mathrm{B}:\alpha_{n+1}\leq\alpha_n,\ (q*\alpha)_n=O(\alpha_n(1*q)_n),\ (N,q,\alpha)\ \textit{is regular};$$

holds. Then

$$(\overline{N},\alpha)\Longrightarrow(N,q).$$

We remark that if $\alpha_n\to\alpha$ as $n\to\infty$, then the regularity of (N,q,α) method from Case B of Theorem 3′ may be omitted as it is implied by other hypotheses. Since $\{q_n\}\in\mathfrak{M}$, therefore, for ν fixed,

$$\frac{q_{n-\nu}}{(q*\alpha)_n}\leq\frac{q_{n-\nu}}{\alpha_n(1*q)_n}\leq\frac{q_{n-\nu}}{\alpha(1*q)_n}\leq\frac{1}{\alpha(n-\nu+1)}=o(1).$$

Compare Theorems 3′ with Theorem 1(c) of [4] where $\{q_n\}$ has been assumed to be increasing.

Also Theorems 1(a) and 1(b) of [4] can easily be deduced from Theorem 3.

It is worth noting that from Theorem 1 of [4] we can deduce Hardy's Theorem [5, Theorem 23] for the case $n_0=0$ only.

But Hardy's Theorem is completely deducible from our Theorem 4. Also putting $p_n=q_n=1$ in Corollay 1, we obtain Cesaro's Theorem [5, Theorem 14].

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Indian Institute of Technology
Hauz Khas, New Delhi—110029
India

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