Kyungpook Math. J.
Volume 16, Number 1
June, 1976

A RESULT ON AN EXTENSION OF FOX'S H-FUNCTION

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1. Introduction

Recently, Agarwal and Mathur [(1), p.536] has given an extension of Fox's H-function [(4), p.408] in two variables by means of a double Mellin-Barnes contour integral in the form

$$(1.1)\qquad H\begin{bmatrix} x \\ y \end{bmatrix} \equiv H^{n,\nu_1,\nu_2,\quad m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{array}{c|c} x & \begin{array}{c}\{(\varepsilon_p, e_p)\}\\ \{(\gamma_t, c_t)\};\ \{(\gamma'_{t'}, c'_{t'})\}\end{array} \\ y & \begin{array}{c}\{(\delta_s, d_s)\}\\ \{(\beta_q, b_q)\};\ \{(\beta'_{q'}, b'_{q'})\}\end{array}\end{array}\right]$$

$$= \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\phi(\xi+\eta)\phi(\xi,\eta)x^{\xi}y^{\eta}d\xi\ d\eta$$

where (i) $\{(\varepsilon_p, e_p)\}$ stands for $(\varepsilon_1, e_1), \cdots, (\varepsilon_p, e_p)$ and similarly for $\{(\gamma_t, c_t)\}$, $\{(\gamma'_{t'}, c'_{t'})\}$ etc.,

(ii) $$\phi(\xi+\eta) = \frac{\prod_{j=1}^{n}\Gamma(1-\varepsilon_j+e_j\xi+e_j\eta)}{\prod_{j=n+1}^{p}\Gamma(\varepsilon_j-e_j\xi-e_j\eta)\prod_{j=1}^{s}\Gamma(\delta_j+d_j\xi+d_j\eta)},$$

(iii) $\phi(\xi, \eta)$

$$= \frac{\prod_{j=1}^{\nu_1}\Gamma(\gamma_j+c_j\xi)\prod_{j=1}^{m_1}\Gamma(\beta_j-b_j\xi)\prod_{j=1}^{\nu_2}\Gamma(\gamma'_j+c'_j\eta)\prod_{j=1}^{m_2}\Gamma(\beta'_j-b'_j\eta)}{\prod_{j=\nu_1+1}^{t}\Gamma(1-\gamma_j-c_j\xi)\prod_{j=m_1+1}^{q}\Gamma(1-\beta_j+b_j\xi)\prod_{j=\nu_2+1}^{t'}\Gamma(1-\gamma'_j-c'_j\eta)\prod_{j=m_2+1}^{q'}\Gamma(1-\beta'_j+b'_j\eta)}$$

and

(iv) $0\le n\le p,\ 0\le\nu_1\le t,\ 0\le\nu_2\le t',\ 0\le m_1\le q,\ 0\le m_2\le q'$.

The sequences of parameters $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p$; $\gamma_1, \gamma_2, \cdots, \gamma_{\nu_1}$; $\beta_1, \beta_2, \cdots, \beta_{m_1}$; $\gamma'_1, \gamma'_2, \cdots, \gamma'_{\nu_2}$; and $\beta'_1, \beta'_2, \cdots, \beta'_{m_2}$ are such that none of the poles of the integrand coincide, the paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma(\beta_j-b_j\xi)$, $j=1,2,\cdots, m_1$ and $\Gamma(\beta'_j-b'_j\eta)$, $j=1,2,\cdots, m_2$ lie to the right and those of $\Gamma(\gamma_j+c_j\xi)$, $j=1,2,\cdots,\nu_1$, $\Gamma(\gamma'_j+c'_j\eta)$, $j=1,2,\cdots\nu_2$ and $\Gamma(1-\varepsilon_j+e_j\xi+e_j\eta)$, $j=1,2,\cdots, n$ lie to the left and the double integral converges if

$$(1.2)\quad \begin{cases} 2(n+\nu_1+m_1)>p+s+t+q, & |\arg(x)|<[n+\nu_1+m_1-\frac{1}{2}(p+s+t+q)]\pi, \\ 2(n+\nu_2+m_2)>p+s+t'+q', & |\arg(y)|<[n+\nu_2+m_2-\frac{1}{2}(p+s+t'+q')]\pi. \end{cases}$$

In this note a finite integral involving the extended H-function in two arguments has been evaluated by the method based on interchanging the order of integration, and term-by-term integration with the help of known integral and also obtained its double-integral analogues as well as some interesting particular cases which generalize several recent results of MacRobert [(5)], Shah [(6)-(7)&(8)] and many others in the theory of special functions.

2. The main integral

We begin by considering the integral

$$I=\int_0^u x^{\rho-1}(u-x)^{\sigma-1} H^{n,\nu_1,\nu_2}_{p,[t:t'],s,[q:q']}{}^{m_1,m_2}\left[\begin{matrix}\lambda x^k(u-x)^l \\ \mu x^k(u-x)^l\end{matrix}\left|\begin{matrix}\{(\varepsilon_p,e_p)\} \\ \{(\gamma_t,c_t)\};\ \{(\gamma'_{t'},c_{t'})\} \\ \{(\delta_s,d_s)\} \\ \{(\beta_q,b_q)\};\ \{(\beta'_{q'},b'_{q'})\}\end{matrix}\right.\right]dx$$

where k and l are to be non-negative integers both not zero.

On substituting $x=uz$ and then the value of $H\left[\begin{matrix}u^{(k+l)}z^k(1-z)^l \\ u^{(k+l)}z^k(1-z)^l\end{matrix}\right]$ in the integrand from (1.1), if we invert the order of integration, which can easily be justified by De La Vallée Poussin's theorem [(2), p. 504] in view of conditions stated in (1.2) earlier, and evaluate the inner z-integral by making use of the known β-integral [(3), p. 425] and then by virtue of (1.1), we shall obtain

$$(2.1)\quad I=u^{\rho+\sigma-1} H^{n+2,\nu_1,\nu_2}_{p+2,[t:t'],s,[q:q']}{}^{m_1,m_2}\left[\begin{matrix}\lambda u^{(k+l)} \\ \mu u^{(k+l)}\end{matrix}\left|\begin{matrix}(1-\rho,k),(1-\sigma,l),\{(\varepsilon_p,e_p)\} \\ \{(\gamma_t,c_t)\};\ \{(\gamma'_{t'},c'_{t'})\} \\ (\rho+\sigma,k+l),\ \{(\delta_s,d_s)\} \\ \{(\beta_q,b_q)\};\ \{(\beta'_{q'},b'_{q'})\}\end{matrix}\right.\right]$$

provided

$$\text{Re}\left[\rho+(k+l)\left(\frac{\beta_j}{b_j}+\frac{\beta'_i}{b'_i}\right)\right]>0,\ \text{and Re}\left[\sigma+(k+l)\left(\frac{\beta_j}{b_j}+\frac{\beta'_i}{b'_i}\right)\right]>0,$$

$j=1,2,\cdots,m_1;\ i=1,2,\cdots,m_2$

and either (1.2) [with x and y replaced by λ and μ] holds, or

$$p+t<s+q,\quad p+t'<s+q',$$

or else $p+t=s+q, p+t'=s+q'$ with $|\lambda|<1$ and $|\mu|<1$.

3. Particular cases

(1) For $p=s=n=0$, the double integral in (1.1) breaks up into the product of two Fox's H-functions

$$H_{t,\ q}^{m_1,\nu_1}\left[x\left|\begin{matrix}\{(1-\gamma_t,\ c_t)\}\\ \{(\beta_q,b_q)\}\end{matrix}\right.\right]H_{t',q'}^{m_2,\nu_2}\left[y\left|\begin{matrix}\{(1-\gamma'_{t'},\ c'_{t'})\}\\ \{(\beta'_{q'},b'_{q'})\}\end{matrix}\right.\right]$$

where $q\geq t, q'\geq t'$ and from (2.1), we thus obtain

$$(3.1)\ \int_0^u x^{\rho-1}(u-x)^{\sigma-1}H_{t,\ q}^{m_1,\nu_1}\left[\lambda x^k(u-x)^l\left|\begin{matrix}\{(\gamma_t,\ c_t)\}\\ \{(\beta_q,b_q)\}\end{matrix}\right.\right]H_{t',\ q'}^{m_2,\nu_2}\left[\mu x^k(u-x)^l\left|\begin{matrix}\{(\gamma'_{t'},\ c'_{t'})\}\\ \{(\beta'_{q'},b'_{q'})\}\end{matrix}\right.\right]dx$$

$$=u^{\rho+\sigma-1}H_{2,\ [t\ :\ t'],1,\ [q\ :\ q']}^{2,\nu_1,\nu_2,\qquad m_1,m_2}\left[\begin{matrix}\lambda u^{(k+l)}\\ \mu u^{(k+l)}\end{matrix}\left|\begin{matrix}(1-\rho,k),\ (1-\sigma,l)\\ \{(1-\gamma_t,\ c_t)\};\ \ \{(1-\gamma'_{t'},\ c'_{t'})\}\\ (\rho+\sigma,k+l)\\ \{(\beta_q,b_q)\};\ \{(\beta'_{q'},b'_{q'})\}\end{matrix}\right.\right]$$

which holds under the same conditions as stated in (2.1) with $p=s=n=0$.

Further on substituting $k=0$ or $l=0$ in (3.1), we can obtain the result in the simplest modified form.

(2) On the other hand, since

$$\lim_{y\to 0}H_{p,\ [t\ :\ 0],s,\ [q\ :\ 1]}^{p,\nu_1,0,\qquad m_1,1}\left[\begin{matrix}x\\ y\end{matrix}\left|\begin{matrix}\{(\varepsilon_p,e_p)\}\\ \{(\gamma_t,\ c_t)\};-\\ \{(\delta_s,d_s)\}\\ \{(\beta_q,b_q)\};\ 0\end{matrix}\right.\right]=H_{p+t,s+q}^{m_1,p+\nu_1}\left[x\left|\begin{matrix}\{(\varepsilon_p,e_p)\},\ \ \{(1-\gamma_t,\ c_t)\}\\ \{(\beta_q,b_q)\},\ \ \{(1-\delta_s,d_s)\}\end{matrix}\right.\right]$$

the special case $p=t'=n=\nu_2=m_2-1=q'-1=0$ and then replacing $p+\nu_1$, $p+t$ and $s+q$ by ν_1, t and q respectively of (2.1) yields the elegant formula

$$(3.2)\ \int_0^u x^{\rho-1}\ (u-x)^{\sigma-1}\ H_{t,\ q}^{m_1,\nu_1}\left[\lambda x^k(u-x)^l\left|\begin{matrix}\{(\gamma_t,\ c_t)\}\\ \{(\beta_q,b_q)\}\end{matrix}\right.\right]dx$$

$$=u^{\rho+\sigma-1}\ H_{t+2,q+1}^{m_1,\nu_1+2}\left[\lambda u^{(k+l)}\left|\begin{matrix}(1-\rho,k),(1-\sigma,l),\{(\gamma_t,\ c_t)\}\\ \{(\beta_q,b_q)\},(1-\rho-\sigma,k+l)\end{matrix}\right.\right]$$

where, for convergence, $2(m_1+\nu_1)>t+q$, $|\arg(\lambda)|<\left[m_1+\nu_1-\frac{1}{2}(t+q)\right]\pi$, $\mathrm{Re}\left[\rho+(k+l)\frac{\beta_j}{b_j}\right]>0$, $j=1,2,\cdots,\ m_1$ and k and l are positive integers.

Possible deductions are:

(a) Setting $l=0$, all c's and b's and k equal to unity, a known result [(3), p. 417, (1)] can be obtained.

(b) Substituting $l=0$, all c's and b's equal to unity, $m_1=t, \nu_1=1$, $t=q+1$, $q=t$, $\gamma_1=1$ and using the known relation [(3), p. 444]:

$$G^{t,\ 1}_{q+1,t}\left(x\left|\begin{matrix}1,\gamma_q\\ \beta_t\end{matrix}\right.\right)=E(t:\beta_p:q:\gamma_r:x),$$

we can obtain a well-known interesting result due to MacRobert [(5), p. 235].

(c) With $l=0,\ m_1=q=4,\ \nu_1=0,\ t=2,\ \gamma_1=\frac{1}{2}+a,\ \gamma_2=\frac{1}{2}-a,\ \beta_1=0,\ \beta_2=\frac{1}{2},$ $\beta_3=b,\ \beta_4=-b,\ c_1=c_2=b_1=b_2=b_3=b_4=1$, and by virtue of the known relation [(3), p. 438], we get

$$(3.3)\quad \int_0^u x^{\rho-\frac{1}{2}k-1}(u-x)^{\sigma-1}W_{a,b}(2\sqrt{\lambda x^k})\ W_{-a,b}(2\sqrt{\lambda x^k})dx$$
$$=\sqrt{\frac{\lambda}{\pi}}u^{\rho+\sigma-1}\Gamma(\sigma)H^{4,1}_{3,5}\left[\lambda u^k\left|\begin{matrix}(1-\rho,k),\ \left(\frac{1}{2}+a,1\right),\ \left(\frac{1}{2}-a,1\right)\\ (0,1),\ \left(\frac{1}{2},1\right),\ (b,1),\ (-b,1),\ (1-\rho-\sigma,k)\end{matrix}\right.\right]$$

where $|\arg(x)|<\pi$, $\mathrm{Re}[\rho+k(\pm b)]>0$, k is a positive integer>0, and $W_{\pm a,b}\ (x)$ are Whittaker functions.

(d) Taking $l=0,\ m_1=2,\ q=3,\ \nu_1=t=1,\ \gamma_1=\frac{1}{2},\ \beta_1=a,\ \beta_2=0,\ \beta_3=-a,\ c_1=b_1$ $=b_2=b_3=1$ and with the application of the relation [(3), p. 437], we have

$$(3.4)\quad \int_0^u x^{\rho-1}(u-x)^{\sigma-1}\ I_a(\sqrt{\lambda x^k})K_a(\sqrt{\lambda x^k})dx$$
$$=\frac{u^{\rho+\sigma-1}\Gamma(\sigma)}{2\sqrt{\pi}}H^{2,2}_{2,4}\left[\lambda u^k\left|\begin{matrix}(1-\rho,k),\ \left(\frac{1}{2},1\right)\\ (a,1),\ (0,1),\ (-a,1),\ (1-\rho-\sigma,k)\end{matrix}\right.\right]$$

where k is a positive integer>0, $|\arg(\lambda)|<\pi$, $\mathrm{Re}[\rho+k(\pm a)]>0$, $I_a\ (x)$ and $K_a\ (x)$ are modified-functions.

4. The double-integral analogues

Adopting the same procedure as above, we can derive in a rather straight forward manner the double-integral analogues for the extended H-function in two arguments as

$$(4.1)\quad \int_0^u\int_0^v x^{\rho-1}(u-x)^{\sigma-1}y^{\alpha-1}(v-y)^{\delta-1}$$
$$\times H^{n,\nu_1,\nu_2,\quad m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix}\lambda x^k(u-x)^l\\ \mu y^h(v-y)^r\end{matrix}\left|\begin{matrix}\{(\varepsilon_p,e_p)\}\\ \{(\gamma_t,c_t)\};\ \{(\gamma'_{t'},c'_{t'})\}\\ \{(\delta_s,d_s)\}\\ \{(\beta_q,b_q)\};\ \{(\beta'_{q'},b'_{q'})\}\end{matrix}\right.\right]dxdy$$
$$=u^{\rho+\sigma-1}v^{\alpha+\delta-1}H^{n,\nu_1+2,\nu_2+2,\quad m_1,m_2}_{p,[t+2:t'+2],s,[q+1:q'+1]}$$
$$\times\left[\begin{matrix}\lambda u^{(k+l)}\\ \mu v^{(h+r)}\end{matrix}\left|\begin{matrix}\{(\varepsilon_p,e_p)\}\\ (\rho,k),\ (\sigma,l),\ \{(\gamma_t,c_t)\};\ (\alpha,h),\ (\delta,r),\ \{(\gamma'_{t'},c'_{t'})\}\\ \{(\delta_s,d_s)\}\\ \{(\beta_q,b_q)\},\ (1-\rho-\sigma,k+l);\ \{(\beta'_{q'},b'_{q'})\},\ (1-\alpha-\delta,h+r)\end{matrix}\right.\right]$$

where k, l, h and r are to be non-negative integers and set of the conditions of the validity:

$$\text{(i)}\ \begin{cases} 2(n+\nu_1+m_1)>p+s+t+q, & |\arg(\lambda)|<\left[[n+\nu_1+m_1-\frac{1}{2}(p+s+t+q)\right]\pi \\ 2(n+\nu_2+m_2)>p+s+t'+q', & |\arg(\mu)|<\left[n+\nu_2+m_2-\frac{1}{2}(p+s+t'+q')\right]\pi \\ \mathrm{Re}\left[\rho+(k+l)\left(\frac{\beta_j}{b_j}+\frac{\beta'_i}{b'_i}\right)\right]>0, & \mathrm{Re}\left[\sigma+(k+l)\left(\frac{\beta_j}{b_j}+\frac{\beta'_i}{b'_i}\right)\right]>0, \\ \mathrm{Re}\left[\alpha+(h+r)\left(\frac{\beta_j}{b_j}+\frac{\beta'_i}{b'_i}\right)\right]>0, & \mathrm{Re}\left]\delta+(h+r)\left(\frac{\beta_j}{b_j}+\frac{\beta'_i}{b'_i}\right)\right]>0, \\ j=1,2,\cdots,\ m_1 \text{ and } i=1,2,\cdots,\ m_2, & \end{cases}$$

$$\text{(ii)}\ \begin{cases} p+t<s+q,\ p+t'<s+q' \text{ or else } p+t=s+q,\ p+t'=s+q' \text{ with } |\lambda|<1,\ |\mu|<1, \\ \mathrm{Re}\left[\rho+(k+l)\left(\frac{\beta_j}{b_j}+\frac{\beta'_i}{b'_i}\right)\right]>0, \quad \mathrm{Re}\left[\sigma+(k+l)\left(\frac{\beta_j}{b_j}+\frac{\beta'_i}{b'_i}\right)\right]>0, \\ \mathrm{Re}\left[\alpha+(h+r)\left(\frac{\beta_j}{b_j}+\frac{\beta'_i}{b'_i}\right)\right]>0, \quad \mathrm{Re}\left[\delta+(h+r)\left(\frac{\beta_j}{b_j}+\frac{\beta'_i}{b'_i}\right)\right]>0, \\ j=1,2,\cdots,\ m_1;\ i=1,2,\cdots,\ m_2. \end{cases}$$

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