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ON THE CATEGORY OF QUASI-UNIFORM SPACES

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1. Introduction

The category of topological spaces has recently attracted a great deal of attention. Herrlich and Strecker in [2] have considered coreflective subcategories in the category of topological spaces. Applying their techniques, we are able to characterize the coreflective subcategories in the category of quasi-uniform spaces. It is noted that the category of topological spaces is a retract of the category of quasi-uniform spaces and it is shown that the category of uniform spaces is coreflective in the category of quasi-uniform spaces.

DEFINITION 1.1. A *quasi-uniform structure* $\mathscr{U}$ for a nonempty set X is a filter on $X\times X$ satisfying:

(1) $\Delta=\{(x,x): x\in X\}\subset U$ for each U in $\mathscr{U}$,

(2) for each U in $\mathscr{U}$ there exists a V in $\mathscr{U}$ with $V\circ V\subset U$.

DEFINITION 1.2. Let $\mathscr{U}$ be a quasi-uniform structure on X. Then let $t_{\mathscr{U}}=\{O\subset X$: if $x\in O$ then there exists U in $\mathscr{U}$ with $x\in U[x]\subset O\}$.

It is easy to show that $t_{\mathscr{U}}$ is a topology on X. A quasi-uniform structure $\mathscr{U}$ on X is said to be *compatible* with a topology t on X if $t=t_{\mathscr{U}}$.

In [4], Pervin showed that the collection $S=\{O\times O\cup(X-O)\times X: O\in t\}$ formed a subbase for a quasi-uniform structure for a topological space (X,t) which is compatible with t. An excellent introduction to quasi-uniform spaces may be found in [3].

2. Category of quasi-uniform spaces

Let $\mathscr{Q}$ denote the category of quasi-uniform spaces and quasi-uniformly continuous maps. $\mathscr{Q}'$ will denote the category of nonempty quasi-uniform spaces.

THEOREM 2.1. *In the category* $\mathscr{Q}$,

(1) *a morphism is a monomorphism if and only if it is one-to-one,*

(2) *a morphism is an epimorphism if and only if it is surjective,*

(3) *an isomorphism is a quasi-uniform space isomorphism,*

(4) *products are the quasi-uniform space products,*

(5) *coproducts are the disjoint quasi-uniform space union.*

For a given set X we let $\mathscr{U}_{X\times X}=\{X\times X\}$ and $\mathscr{U}_{\Delta}=\{U\subset X\times X: \Delta\subset U\}$.

THEOREM 2.2. *In the category* $\mathscr{Q}$,

(1) *the only initial object is* $(\phi, \mathscr{U}_{\phi\times\phi})$

(2) *the terminal objects are of the form* $(\{a\}, \mathscr{U}_{\Delta})$,

(3) *has no zero object,*

(4) *t'e injective objects are precisely quasi-uniform spaces of the form* $(X, \mathscr{U}_{X\times X})$,

(5) *the projective objectives are precisely quasi-uniform spaces of the form* $(X, \mathscr{U}_{\Delta})$.

Let f: $(X, \mathscr{U})\to Y$ be surjective and set $\mathscr{V}$ equal to the supremum of all quasi-uniform structures on Y for which f is quasi-uniformly continuous. Then $f:(X, \mathscr{U})\to(Y, \mathscr{V})$ is called a quotient map.

THEOREM 2.3. *In the category* $\mathscr{Q}$,

(1) $f:(X, \mathscr{U})\to(Y, \mathscr{W})$ *is an extremal monomorphism if and only if* $(Y, \mathscr{U})$ *is quasi-uniformly isomorphic to the subspace* $f(X)$.

(2) $q:(X, \mathscr{U})\to(Y, \mathscr{W})$ *is an extremal epimorphism if and only if* $(Y, \mathscr{W})$ *is quasi-uniformly isomorphic to* $(Y, \mathscr{V})$ *where* $\mathscr{V}$ *is the quotient structure induced by* q.

This theorem shows that the extremal monomorphisms in $\mathscr{Q}$ are precisely the embedding maps while the extremal epimorphisms are the quotient maps.

Consider the morphism $f:(X, \mathscr{U})\to(Y, \mathscr{V})$. Let $Z=f(X)$ and $\mathscr{W}$ be the restriction of $\mathscr{V}$ to $f(X)$. Let $f:X\to Z$ be defined by $f(x)=f(x)$ for each x in X. Let $i:Z\to Y$ be the identity mpping. Then $f=if$ and $\mathscr{Q}$ has the epi-mono factorization property. Moreover, if we let $\mathscr{W}$ be the quotient structure on $f(X)$ then f is an extremal epimorphism and $\mathscr{Q}$ thus has the extremal epi-mono factorization property. Since the category $\mathscr{Q}$ is locally small we have by theorem 3 in [1] that $\mathscr{Q}$ has the unique extremal epi-mono factorization property.

THEOREM 2.4. *The composite of two extremal epimorphisms in* $\mathscr{Q}$ *is an extremal epimorphism. Thus* $\mathscr{Q}$ *has the strong unique epi-mono factorization property.*

PROOF. Let $(X,\mathscr{U})\xrightarrow{f}(Y,\mathscr{V})\xrightarrow{g}(Z,\mathscr{W})$ be given where f and g are extremal epimorphisms. Let $\mathscr{S}$ be the supremum of all quasi-uniform structures on Z for which gf is quasi-uniformly continuous. Since gf is quasi-uniformly continuous with respect to $\mathscr{W}$, we have that $\mathscr{W}\leq\mathscr{S}$. Now consider:

$$\begin{array}{ccccc} (X,\mathscr{U}) & \xrightarrow{f} & (Y,\mathscr{V}) & \xrightarrow{g} & (Z,\mathscr{W}) \\ & {\scriptstyle gf}\searrow & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle i} & \\ & & (Z,\mathscr{S}) & & \end{array}$$

i denotes the identity map and is quasi-uniformly continuous since $\mathscr{W}\leq\mathscr{S}$. $g:(Y,\mathscr{V})\to(Z,\mathscr{S})$ is quasi-uniformly continuous since $g^{-1}(\mathscr{V})$ is a quasi-uniform structure on Y for which f is quasi-uniformly continuous. This follows from the fact that f is an extremal epimorphism and $\mathscr{V}$ is the strongest quasi-uniform structure on Y for which f is quasi-uniformly continuous. Since g: $(Y,\mathscr{V})\to(Z,\mathscr{W})$ is an extremal epimorphism and $g=ig$ where i is a monomorphism, we must have that i: $(Z,\mathscr{S})\to(Z,\mathscr{W})$ is an isomorphism. Thus $\mathscr{S}\leq\mathscr{W}$ and hence $\mathscr{S}=\mathscr{W}$. Therefore gf is an extremal epimorphism.

Now since $\mathscr{Q}$ has the unique extremal epi-mono factorization property and the composite of extremal epimorphisms is an extremal epimorphism we have that $\mathscr{Q}$ has the strong unique extremal epi-mono factorization property.

THEOREM 2.5. *The constant morphisms in $\mathscr{Q}$ are precisely the constant maps. The category $\mathscr{Q}'$, of nonempty quasi-uniform spaces, is constant generated.*

PROOF. The first statement is evident. Let $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ be objects in $\mathscr{Q}'$. Since Y is nonempty, there exists an element y in Y. Define f: $X\to Y$ by $f(x)=y$ for each x in X. Thus the set of morphisms from X to Y is nonempty.

Now let f,g: $X\to Y$ be distinct morphisms. Then there is an element x in X with $f(x)\neq g(x)$. Set $Z=\{x\}$ and k: $(Z,\mathscr{U}_{\Delta})\to(X,\mathscr{U})$ defined by $k(x)=x$ is a constant morphism such that $fk\neq gk$. Hence $\mathscr{Q}'$ is constant generated.

3. Coreflective subcategories

In this section each subcategory considered is assumed to be nontrivial. A subcategory $\mathscr{U}$ of a categry $\mathscr{C}$ is said to be coreflective in $\mathscr{C}$ if for each object X in $\mathscr{C}$ there exists an object $X_{\mathscr{U}}$ in $\mathscr{U}$ and a morphism $c_{\mathscr{U}}: X_{\mathscr{U}}\to X$, called the coreflective morphisms, such that for each object B in $\mathscr{U}$ and morphism $g:B\to X$ there exists a unique morphism h: $B\to X_{\mathscr{U}}$ such that $g=c_{\mathscr{U}}h$. $\mathscr{U}$ is called epicoreflective if additionally each coreflective morphism is an epimorphism and it is called mono-coreflective if each coreflective morphism is a monomorphism.

For the convience of the reader we state the following theorems found in [1].

THEOREM A. *If $\mathscr{U}$ is a coreflective subcategory of a constant generated category $\mathscr{C}$ then $\mathscr{U}$ is both mono-coreflective and epi-coreflective.*

THEOREM B. *If $\mathscr{C}$ is a category which is*

(a) *locally small,*

(b) *has products,*

(c) *has the extremal epi-mono factorization property,*

and if $\mathscr{U}$ is a subcategory of $\mathscr{C}$ then the following statements are equivalent.

(1) *$\mathscr{U}$ is mono-coreflective in $\mathscr{C}$.*

(2) *$\mathscr{U}$ is closed under the formation of coproducts and extremal quotient objects.*

THEOREM 3.1. *Let $\mathscr{U}$ be a subcategory of $\mathscr{Q}$. The following statements are equivalent.*

(1) *$\mathscr{U}$ is coreflective in $\mathscr{Q}$,*

(2) *$\mathscr{U}$ is mono-coreflective and epi-coreflective in $\mathscr{Q}$,*

(3) *$\mathscr{U}$ is closed under the formation of disjoint unions and quotient objects.*

PROOF. (1)$\Longleftrightarrow$(2) Let $\mathscr{U}$ be a coreflective subcategory of $\mathscr{Q}$. Since we are considering only nontrivial subcategories we have that $\mathscr{U}$ is coreflective in $\mathscr{Q} \Longleftrightarrow \mathscr{U} \cap \mathscr{Q}'$ is coreflective in $\mathscr{Q}'$. Since $\mathscr{Q}'$ is constant generated by theorem 2.5, we have by theorem A that each coreflective subcategory of $\mathscr{Q}'$ must be both mono-coreflective and epi-coreflective. Hence each coreflective morphism $c_{\mathscr{U}}: X_{\mathscr{U}} \rightarrow X$ is one-to-one and onto.

(2)$\Longleftrightarrow$(3) Since $\mathscr{Q}$ satisfies the hypothesis for theorem B we have that a subcategory $\mathscr{U}$ of $\mathscr{Q}$ is mono-coreflective if and only if (3) is satisfied, but if $\mathscr{U}$ is mono-coreflective then it is epi-coreflective by (1)$\Longrightarrow$(2).

We now establish that for each subcategory $\mathscr{U}$ of $\mathscr{Q}$ there exists a smallest coreflective subcategory $\mathscr{B}(\mathscr{U})$ containing $\mathscr{U}$ and moreover that the objects of $\mathscr{B}(\mathscr{U})$ are precisely the quotient objects of disjoint unions of members of $\mathscr{U}$. The following theorems are found in [1].

THEOREM C. *If $\mathscr{C}$ is a category which is*

(1) *locally small,*

(2) *has coproducts, and*

(3) *has the extremal epi-mono factorization property*

and if $\mathscr{U}$ is a subcategory of $\mathscr{C}$ then there exists a smallest mono-coreflective *subcategory $\mathscr{B}$ of $\mathscr{C}$ containing $\mathscr{U}$. Furthermore, if $\mathscr{C}$ has the strong unique extremal epi-mono factorization property then the objects of $\mathscr{B}$ are exactly all extremal quotient objects of coproducts of objects in $\mathscr{U}$*

Let $\mathscr{C}(\mathscr{U})$ denote the smallest mono-coreflective subcategory in the category $\mathscr{C}$ containing the subcategory $\mathscr{U}$.

THEOREM D. *If $\mathscr{C}$ is a category which*

(1) *is locally small,*

(2) *has coproducts,*

(3) *has the strong unique extremal epi-mono factorization property,*

and if $\mathscr{U}$ is any subcategory of $\mathscr{C}$ then each monomorphism in $\mathscr{C}$ which is $\mathscr{U}$-liftable is also $\mathscr{C}(\mathscr{U})$-liftable.

Using theorems C and D together with the fact that each coreflective subcategory in $\mathscr{Q}$ is mono-coreflective we have the following theorem.

THEOREM 3.2. *Let $\mathscr{U}$ be and subcategory of $\mathscr{Q}$. Then*

(1) *there exists a smallest coreflective subategory $\mathscr{B}(\mathscr{U})$ containing $\mathscr{U}$,*

(2) *objects of $\mathscr{B}(\mathscr{U})$ are precisely the quotient objects of disjoint unions of objects in $\mathscr{U}$,*

(3) *each monomorphism in $\mathscr{Q}$ which is $\mathscr{U}$-liftable is $\mathscr{B}(\mathscr{U})$-liftable.*

We now consider some interesting subcategories of $\mathscr{Q}$ that are coreflective in $\mathscr{Q}$.

THEOREM 3.3. *The category of uniform spaces is a coreflective subcategory in $\mathscr{Q}$, the category of quasi-uniform spaces.*

PROOF. Let $(X,\mathscr{U})$ be a quasi-uniform space. Now $(X,\mathscr{U}\vee\mathscr{U}^{-1})$ is a uniform space and the identity map i: $(X,\mathscr{U}\vee\mathscr{U}^{-1})\to(X,\mathscr{U})$ is quasi-uniformly continuous. Let $(Y,\mathscr{V})$ be any uniform space and f a morphism from $(Y,\mathscr{V})$ to $(X,\mathscr{U})$. Define $\tilde{f}:(Y,\mathscr{V})\to(X,\mathscr{U}\vee\mathscr{U}^{-1})$ by $\tilde{f}(y)=f(y)$ for each y in Y. Now $f=i\tilde{f}$ and $\tilde{f}$ is unique. We must show that $\tilde{f}$: $(Y,\mathscr{V})\to(X,\mathscr{U}\vee\mathscr{U}^{-1})$ is quasi-uniformly continuous. It suffices to show that $\tilde{f}^{-1}(U\cap U^{-1})\in\mathscr{V}$ for each $U\in\mathscr{U}$. Let $U\in\mathscr{U}$, then $f^{-1}(U)\in\mathscr{V}$ and hence $\tilde{f}^{-1}(U)\in\mathscr{V}$. Since $\mathscr{V}$ is a uniform structure, there exists a symmetric $V\in\mathscr{V}$ with $V\subset\tilde{f}^{-1}(U)$. Thus $V\subset\tilde{f}^{-1}(U^{-1})$ and $\tilde{f}^{-1}(U\cap U^{-1})\in\mathscr{V}$. Hence f is quasi-uniformly continuous.

THEOREM 3.4. *The category of fine quasi-uniform spaces is coreflective in the category of quasi-uniform spaces.*

The proof of this theorem is natural. A quasi-uniform space $(X, \mathscr{U})$ is called saturated if $\mathscr{U}$ is closed under arbitrary intersections. A space is saturated if and only if the structure $\mathscr{U}$ has a base consisting of a single set.

THEOREM 3.5. *The category of saturated quasi-uniform spaces is coreflective in $\mathscr{Q}$.*

PROOF. Let $(X, \mathscr{U})$ be a quasi-uniform space. Set $S=\cap\{U : U\in\mathscr{U}\}$. Then $S\circ S=S$ and $\{S\}$ forms a base for a saturated quasi-uniform structure $\mathscr{S}$. Let $i:(X, \mathscr{S})\to(X, \mathscr{U})$ denote the identity map, then i is quasi-uniformly continuous. Suppose that $(Y, \mathscr{V})$ is a saturated space and $f:(Y, \mathscr{V})\to(X, \mathscr{U})$ a morphism in $\mathscr{Q}$. Now define $\tilde{f}:(Y, \mathscr{V})\to(X, \mathscr{S})$ by $\tilde{f}(y)=f(y)$ for each y in Y. Then $f=i\tilde{f}$ and $\tilde{f}$ is unique. To see that $\tilde{f}$ is quasi-uniformly continuous, note that $\mathscr{V}$ is generated by a base $\{T\}$. Then for each U in $\mathscr{U}$ we have $T\subset f^{-1}(U)$ and thus $T\subset\tilde{f}^{-1}(U)$. Therefore $T\subset\cap\{\tilde{f}^{-1}(U) : U\in\mathscr{U}\}=\tilde{f}^{-1}(\cap\{U : U\in\mathscr{U}\})=f^{-1}(S)$. Hence $\tilde{f}$ is quasi-uniformly continuous.

4. Special functors

In this section we consider two natural functors.

THEOREM 4.1. *The category of topological spaces is a retract of the category $\mathscr{Q}$, the category of quasi-uniform spaces.*

PROOF. Let $\mathscr{P}$ denote the subcategory of $\mathscr{Q}$ of quasi-uniform spaces with the Pervin quasi-uniform structure. $\mathscr{T}$ will denote the category of topological spaces and continuous maps. Let T: $\mathscr{Q}\to\mathscr{T}$ be the natural functor from a quasi-uniform space to the underlying topological space. Let P: $\mathscr{T}\to\mathscr{P}$ be the functor that associates with each topological space the corresponding Pervin quasi-uniform space. Now $\mathscr{P}$ is a full subcategory of $\mathscr{Q}$, and the functor PT: $\mathscr{Q}\to\mathscr{P}$ is the identity functor on $\mathscr{P}$. Also TP: $\mathscr{T}\to\mathscr{T}$ is the identity functor on $\mathscr{T}$. Hence $\mathscr{P}$, a full subcategory of $\mathscr{Q}$, is a retract of $\mathscr{Q}$ and $\mathscr{P}$ and $\mathscr{T}$ are isomorphic.

Define R: $\mathscr{Q}\to\mathscr{Q}$ by $(X, \mathscr{U})\to(X, \mathscr{U}^{-1})$. If $f:(X, \mathscr{U})\to(Y, \mathscr{V})$ is a morphism in $\mathscr{Q}$ then define $R(f)(x)=f(x)$ for each x in X. R will be called the conjugate functor on $\mathscr{Q}$.

THEOREM 4.2. *R is a functor on $\mathscr{Q}$ such that $R\circ R$ is the identity functor on $\mathscr{Q}$. The fixed points of R are precisely the uniform spaces.*

PROOF. Let $f:(X, \mathscr{U})\to(Y, \mathscr{V})$ be a morphism in $\mathscr{Q}$, and let $V^{-1}\in\mathscr{V}^{-1}$. Then

$V \in \mathscr{V}$ and there exists a $U \in \mathscr{U}$ with $U \subset f^{-1}(V)$. Thus $U^{-1} \subset f^{-1}(V^{-1})$ and f: $(X, \mathscr{U}^{-1}) \rightarrow (Y, \mathscr{V}^{-1})$ is quasi-uniformly continuous. The other properties are easy to verify and R is indeed a functor on $\mathscr{Q}$. That $R \circ R$ is the identity on $\mathscr{Q}$ is evident. Now $R((X, \mathscr{U})) = (X, \mathscr{U})$ if and only if $\mathscr{U} = \mathscr{U}^{-1}$. Thus the fixed points of R are precisely the uniform spaces.

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REFERENCES

[1] H. Herrlich and G. E. Strecker, *Coreflective subcategories*, Trans. Amer. Math. Soc. 157 (1971), 205—226.

[2] H. Herrlich and G. E. Strecker, *Coreflective subcategories in general topology*, Fund. Math. 73 (1972), 199—218.

[3] M. C. Murdeshwar and S. A. Naimpally, *Quasi-Uniform Topological Spaces*, Noordhoff, 1966.

[4] W. J. Pervin, *Quasi-uniformization of topological spaces*, Math. Ann. 147 (1962), 316—317.