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ON T_0' SPACES

By S.M. Lee

In this note we discuss a separation property which is weaker than T_1, but not comparable with T_0. The usage of terminology will be exactly same as in [1]. A topological space $(X,\mathscr{U})$ is said to be a T_0' *space* if for any point not in a closed set A in X, there exists an open set which contains A but not x.

THEOREM 1. *The following statements are equivalent.*

(1) *X is a T_0' space,*

(2) *if $a\in\overline{\{x\}}$ then $\overline{\{a\}}=\overline{\{x\}}$ for any a and x in X,*

(3) *if $x\notin A$ then $\overline{\{x\}}\cap A=\phi$ for any x in X and A closed in X.*

PROOF. Assuming (1) and $a\in\overline{\{x\}}$ it is clear $\overline{\{a\}}\subset\overline{\{x\}}$. If $x\notin\overline{\{a\}}$ then there exists open set $U\supset\overline{\{a\}}$ and $x\notin U$. Therefore $x\in(\overline{\{x\}}\sim U)\subsetneqq\overline{\{x\}}$ and $(\overline{\{x\}}\sim U)$ is closed. This contradicts to the definition of closure. Hence $x\in\overline{\{a\}}$ and $\overline{\{x\}}\subset\overline{\{a\}}$, that implies $\overline{\{x\}}=\overline{\{a\}}$.

Assuming (2), if $\overline{\{x\}}\cap A\neq\phi$, then $a\in\overline{\{x\}}\cap A$, hence $\overline{\{x\}}=\overline{\{a\}}\subset A$ and $x\in A$. This is a contradiction.

Assuming (3) and $x\notin A$ and A is closed, then $X\sim\overline{\{x\}}$ is an open set which contains A but not x.

It is clear that T_1 implies T_0'. However the conditions T_0 and T_0' are independent. The indiscrete space with at least two points is T_0' but not T_0. Let X be the set of real numbers. Let $\mathscr{U}=\{(x:a\leq x<\infty)\,|\,a\in R\}$ Then $(X,\mathscr{U})$ is T_0 but not T_0'. There is a large important class of spaces which are T_0' but not T_1, namely the semi-normed spaces and pseudo-metric spaces. The relation between T_0', T_0 and T_1 is as follows.

THEOREM 2. *A topological spaces is T_1 iff it is T_0' and T_0.*

PROOF. If there is x in X with $\overline{\{x\}}\neq\{x\}$ then there is y, different from x, and $y\in\overline{\{x\}}$. Since X is T_0', by Theorem 1, we have $\overline{\{y\}}=\overline{\{x\}}$, since X is T_0, $y=x$.

This is a contradiction. Therefore $\overline{\{x\}}=\{x\}$ for any x in X.

It is easy to see that the subspace of a T_0' space is T_0'. The next theorem is dealing with the product space of a family of T_0' spaces.

THEOREM 3. *The product of a family of spaces is T_0' iff each member of the family is T_0'.*

PROOF. Let $\{X_\alpha\}_{\alpha\in A}$ be a family of T_0' spaces. Let B be any closed set in $\prod\{X_\alpha\}$ and x a point not in B. If $\overline{\{x\}}\cap B\neq\phi$ then there is a point a in $\overline{\{x\}}\cap B$. But since $\overline{\{x\}}=\overline{\prod\{x_\alpha\}}=\prod\overline{\{x_\alpha\}}$ we have $a_\alpha\in\overline{\{x_\alpha\}}$ for all α in A. Since X_α is T_0' space for each α, by Theorem 1 we have $\overline{\{a_\alpha\}}=\overline{\{x_\alpha\}}$ for all α in A. Hence

$$\overline{\{a\}}=\overline{\prod\{a_\alpha\}}=\prod\overline{\{a_\alpha\}}=\prod\overline{\{x_\alpha\}}=\overline{\prod\{x_\alpha\}}=\overline{\{x\}}$$

Since B is closed and $a\in B$ we have $\overline{\{a\}}\subset B$ hence $\overline{\{x\}}\subset B$ and $x\in B$. This is a contradiction. Hence $\overline{\{x\}}\cap B=\phi$ and $\prod\{X_\alpha\}\sim\overline{\{x\}}$ is open and contains B.

Next suppose that $\prod\{X_\alpha\}$ is T_0' and β be any element in A. If X_β is not T_0', then there is a closed set B_β in X_β and x_β not in B_β such that $B_\beta\cap\overline{\{x_\beta\}}\neq\phi$. Let $p\in\prod\{X_\alpha\}$ such that $p_\beta=x_\beta$. Therefore $p\notin P_\beta^{-1}\{B_\beta\}$ but $\overline{\{p\}}\cap P_\beta^{-1}\{B_\beta\}\neq\phi$ and $P_\beta^{-1}\{B_\beta\}$ is closed that contradicts to that $\prod\{X_\alpha\}$ is T_0'.

Kansas State University
Manhattan, Kansas 66502

REFERENCE

[1] J. Kelley, *General Topology*, Princeton, 1955.