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RIGHT GENERALIZED ω-$\mathscr{L}$-UNIPOTENT BISIMPLE SEMIGROUPS

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Let S be a regular bisimple semigroup such that the union T of the maximal subgroups of S is an ω-chain of rectangular groups (T_n: $n\in N$, the non-negative integers). Also, suppose that $e, f\in E(T)$, the set of idempotents of T, and $ef=e$ imply $gegfe=ge$ for all $g\in E(T)$ and $\mathscr{R}$, Green's relation, is a right congruence on T. We term S a right generalized ω-$\mathscr{L}$-unipotent bisimple semigroup. We characterize such S. Let $(I, \circ)$ be an ω-chain of left zero semigroups (I_n: $n\in N$) and let $(J, \otimes)$ be an ω-chain of right groups (J_n: $n\in N$). Suppose $I_n\cap J_n=\{e_n\}$, a single idempotent element. Let $(n, k)\rightarrow\alpha_{(n,k)}$ be a homomorphism of C, the bicyclic semigroup, into End$(I, \circ)$, the semigroup of endomorphism of $(I, \circ)$ (iteration), and let $(n, k)\rightarrow\beta_{(n,k)}$ be a homomorphism of C into End$(J, \otimes)$ such that 1. $g\beta_{(s,s)}=g\otimes e_s$ for all $g\in J$; 2. $I_r\alpha_{(n,k)}\subset I_{r+k-\min(r,n)}$ and $J_r\beta_{(n,k)}\subset J_{r+k-\min(r,n)}$. Let (I, J, α, β) denote $I\times J$ under the product: if $i\in I_n$, $j\in J_k$, $u\in I_r$, and $v\in J_s$, 3. $(i, j)(u, v)=(i\circ u\alpha_{(k,n)}, j\beta_{(r,s)}\otimes v)$. We show (theorem 2. 12) that (I, J, α, β) is a right generalized ω-$\mathscr{L}$ unipotent bisimple semigroup and, conversely, every such semigroup is isomorphic to some (I, J, α, β).

We use the definitions of Clifford and Preston [1] and of [3, p.102] unless otherwise specified. Particularly, a semigroup S is termed regular if $a\in aSa$ for all $a\in S$. $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, and $\mathscr{D}$ will denote Green's equivalence relations on a semigroup S, i.e., $(a, b)\in R$ if $a\cup aS=b\cup bS$; $(a, b)\in\mathscr{L}$ if $a\cup Sa=b\cup Sb$; $\mathscr{H}=\mathscr{R}\cap\mathscr{L}$; and $\mathscr{D}=\mathscr{R}\circ\mathscr{L}((a, b)\in\mathscr{D}$ if there exists $x\in S$ such that $(a, x)\in\mathscr{R}$ and $(x, b)\in(\mathscr{L})$. Let R_a denote the $\mathscr{R}$-class containing a. A semigroup which contains a single $\mathscr{D}$ class is termed bisimple. A rectangular band is the algebraic direct product of a left zero semigroup $U(x, y\in U$; implies $xy=x)$ and a right zero semigroup. A rectangular group is the algebraic direct product of a group and a rectangular band. A semigroup S which is the union of a collection of pairwise disjoint subsemigroups (S_n: $n\in N$) such that $S_nS_m\subset S_{\max(n,m)}$ is called an ω-chain of the semigroups (S_n: $n\in N$). A semigroup X is called a right group if a, $b\in X$

implies there exists a unique $x \in X$ such that $ax=b$. The bicyclic semigroup is $C=N \times N$ under the product $(m, n)(p, q)=(m+p-\min(n, p), n+q-\min(n, p))$. If V is a subset of a semigroup $S, E(V)$ will denote the set of idempotents of V.

In [2], a regular semigroup S was termed generalized $\mathscr{L}$-unipotent if $e, f \in E(S)$ and $ef=e$ imply $gegfe=ge$ for all $g \in E(S)$.

Let S be a regular bisimple semigroup such that the union T of the maximal subgroups of S is an ω-chain of rectangular groups $(T_n: n \in N)$. Using [2, lemma 1], S is right generalized ω-$\mathscr{L}$-unipotent if and only if $\mathscr{R}$ is a right congruence on $T, E(T)$ is an ω-chain of rectangular bands $(E(T_n): n \in N)$ and $\mathscr{L}$ is a left congruence on $E(T)$. We let $E_n=E(T_n)$.

Using results of [3, section 1], our terminology here is in accordance with that of [3].

1. Structure theorem for right generalized ω-$\mathscr{L}$-unipotent bisimple semigroups (proof of converse)

In this section, S will denote a right generalized ω-$\mathscr{L}$-unipotent bisimple semigroup.

As in [3, notes 1.7 and 1.8], let e_0 be a fixed element of E_0. Let $f \in E_1$ and $e_1=e_0fe_0$. Select and fix $a \in R_{e_0} \cap L_{e_1}$. By the proof of [1, theorem 2.18] there exists a unique inverse a^{-1}(i.e. $aa^{-1}a=a$ and $a^{-1}aa^{-1}=a^{-1}$) of a contained in $R_{e_1} \cap L_{e_0}$ with $aa^{-1}=e_0$ and $a^{-1}a=e_1$. Define $a^{-n}=(a^{-1})^n$ for all positive integers n and define $a^0=e_0$. Let $e_k=a^{-k}a^k$. By [3, note 1.3 and lemma 1.11], $e_k \in E_k$ for each $k \in N$ and $e_ke_n=e_{\max(n,k)}$.

As in [3, paragraph following lemma 1.11], let $J_k(I_k)$ denote the $\mathscr{R}$-class (the set of idempotents of the $\mathscr{L}$-class) of T containing e_k (note, the T's (and T_k's) of [3] are identical to those of this paper by using [3, note 1.3 and proposition 1.4]). Let $I=\cup(I_n: n \in N)$ and $J=\cup(J_n: n \in N)$. By [3, lemma 1.13], J_n is a right group for each $n \in N$.

LEMMA 1.1. [3, lemma 1.12] *I is an ω-chain of left zero semigroups $(I_n: n \in N)$.*

LEMMA 1.2. *J is an ω-chain of right groups $(J_n: n \in N)$.*

PROOF. Let $x \in J_n$ and $y \in J_m$. Hence, $x \mathscr{R} e_n (\in T)$ and $y \mathscr{R} e_m$. Since $\mathscr{R}$ is a congruence on $T, xy \mathscr{R} e_ne_m=e_{\max(n,m)}$. Hence $xy \in J_{\max(n,m)}$.

REMARK. Lemma 1.2 may also be obtained from [3, lemma 1.21].

If X is a set, T_X will denote the semigroup (iteration) of mappings of X into X.

LEMMA 1.3. [3, lemma 1.15] *There exists a mapping* $j\to A_j$ *of* J *into* T_I *and a mapping* $p\to B_p$ *of* I *into* T_J *such that* $I_nA_j\subset I_{\max(n,m)}$ *for* $j\in J_m$ *and* $J_nB_p\subset J_{\max(n,m)}$ *for* $p\in I_m$. *If* $j\in J$ *and* $p\in I$, $jp=pA_jjB_p$. *Furthermore,* $jp\ \mathscr{R}\ pA_j(\in T)$ *and* $jp\ \mathscr{L}\ jB_p(\in T)$.

LEMMA 1.4. $iA_j=e_si$ *for* $j\in J_s$ *and* $i\in I$.

PROOF. First we show that $A_j=A_{e_s}$ for $j\in J_S$. Since $\mathscr{R}$ is a right congruence relation on T, $(j,e_s)\in\mathscr{R}$ implies $(ji,e_si)\in\mathscr{R}$ for all $i\in I$. Hence, using lemma 1.3, $(iA_j,iA_{e_s})\in\mathscr{R}$ for all $i\in I$. Thus, using lemma 1.3, $iA_j=iA_{e_s}$ for all $i\in I$. Let $i\in I_r$, say. Then, since $e_si\in I_{\max(s,r)}$ by lemma 1.1, we utilize lemma 1.3 to obtain $(e_si)e_{\max(s,r)}=e_si=iA_{e_s}e_sB_i$. Therefore, by lemma 1.3 and [3, lemma. 14], $iA_{e_s}=e_si$.

DEFINITION. If $u\in T$ and $n,k\in N$, define $u\nu_{(k,n)}=a^{-n}a^kua^{-k}a^n$. Let $\nu_{(k,n)}|I=\alpha_{(k,n)}$ and $\nu_{(k,n)}|J=\beta_{(k,n)}$.

LEMMA 1.5. [3, lemma 2.6].

a) $I_r\alpha_{(k,n)}\subset I_{r+n-\min(r,k)}$, b) $J_r\beta_{(k,n)}\subset J_{r+n-\min(r,k)}$.

LEMMA 1.6. [3, lemma 2.9] $(k,n)\to\alpha_{(k,n)}$ *is a homomorphism of* C *into* End I.

LEMMA 1.7. $\beta_{(k,n)}\in$End J *for all* $n,k\in N$.

PROOF. By lemma 1.5(b), $\beta_{(k,n)}\in T_J$. Let $g\in J_r$ and $h\in J_s$. Using lemma 1.2, $e_khe_k=e_ke_{\max(k,s)}(he_k)=e_{\max(k,s)}he_k=he_k$. Hence, using [3, lemmas 1.1 and 1.9],

$$\begin{aligned} g\beta_{(k,n)}h\beta_{(k,n)} &= (a^{-n}a^kga^{-k}a^n)(a^{-n}a^kha^{-k}a^n) \\ &= a^{-n}a^kga^{-k}a^kha^{-k}(a^ka^{-k}a^n) \\ &= a^{-n}a^kga^{-k}a^k(ha^{-k}a^k)a^{-k}a^n \\ &= a^{-n}a^kgha^{-k}a^n \\ &= (gh)\beta_{(k,n)}. \end{aligned}$$

REMARK. Lemma 1.7 could also be obtained from [3, lemma 2.15].

LEMMA 1.8. $(k,n)\to\beta_{(k,n)}$ *is a homomorphism of* C *into* End J.

PROOF. Combine lemma 1.7 and [3, lemma 2.12].

LEMMA 1.9. *Let* $\nu \in \text{End } I$ *such that* $e_k \nu \in I_n$. *Then if* $i \in I_n$, $j \in J_k$, *and* $u \in I$, $i(uA_j \nu) = i(u\nu)$.

PROOF. Apply lemma 1.4.

If $a, b \in I$, define $a \circ b = ab$. If $a, b \in J$, define $a \otimes b = ab$.

LEMMA 1.10. $S \cong ((i, (n, k), j)$: $i \in I_n, j \in J_k, n, k \in N)$ *under the multiplication* $(i, (n, k), j)(u, (r, s), v) = (i \circ (u\alpha_{(k,n)}), n + r - \min(k, r), k + s - \min(k, r), j\beta_{(r,s)} \otimes v)$.

PROOF. Use [3, lemma 2.17], definition of "$\circ$", lemma 1.5(a), lemma 1.9, lemma 1.5(b), [3, lemma 1.20], and definition of "$\otimes$".

LEMMA 1.11. $S \cong I \otimes J$ *under the product*: *if* $i \in J_n$, $j \in J_k$, $u \in I_r$, *and* $v \in J_s$, $(i, j)(u, v) = (i \circ u\alpha_{(k,n)}, j\beta_{(r,s)} \otimes v)$.

PROOF. If $i \in I_n$ and $j \in J_k$, $(i, (n, k) j)\phi = (i, j)$ defines an isomorphism of the groupoid given in the statement of lemma 1.10 onto the groupoid given in the statement of lemma 1.11.

LEMMA 1.12. $g\beta_{(s,s)} = g \otimes e_s$.

PROOF. As in the proof of lemma 1.7, $e_s g e_s = g e_s$. Hence, using the definition of $\otimes$, $g\,\beta_{(s,s)} = e_s g e_s = g \otimes e_s$.

REMARK. Lemma 1.12 could also be obtained from [3, lemma 2.19].

THEOREM 1.13. *Let S be a right generalized* ω-$\mathscr{L}$-*unipotent bisimple semigroup. Then S is isomorphic to some* (I, J, α, B).

PROOF. The theorem is a direct consequence of lemmas 1.1, 1.2, 1.5, 1.6, 1.8, 1.11, and 1.12.

2. Structure theorem for right generalized ω-$\mathscr{L}$-unipotent bisimple semigroups (proof of the direct part)

In this section, we show that (I, J, α, β) is a right generalized ω-$\mathscr{L}$-unipotent bisimple semigroup (theorem 2.10).

LEMMA 2.1. (I, J, α, β) *is a semigroup.*

PROOF. Closure follows from the fact that I and J are ω-chains and (2). Let $a = (i\ j), b = (u, v)$, and $c = (w, z)$, where $i \in I_n$, $j \in J_k$, $u \in I_r$, $v \in J_s$, $w \in I_p$, and $z \in J_q$.

Let $a_1=i$ and $a_2=j$. Utilizing the fact that $(n,k)\to\alpha_{(n,k)}$ is a homomorphism of C into End$(I,\circ)$,

$$\begin{aligned}((ab)c)_1&=((i\circ u\alpha_{(k,n)})\circ w\alpha_{(s,r)(k,n)})\\&=(i\circ u\alpha_{(k,n)})\circ w\alpha_{(s,r)}\alpha_{(k,n)}\\&=i\circ((u\circ(w\alpha_{(s,r)}))\alpha_{(k,n)})\\&=(a(bc))_1.\end{aligned}$$

Similarly, using the fact that $(n,k)\to\beta_{(n,k)}$ is a homomorphism of C into End $(J,\otimes)$, $((ab)c)_2=(a(bc))_2$. Hence, $(ab)c=a(bc)$.

LEMMA 2.2. *Let $(i,j),(u,v)\in(I,J,\alpha,\beta)$. Let $i\in I_n$, $j\in J_k$, $u\in I_r$, and $v\in J_s$. Then (a) $(i,j)\mathscr{R}(u,v)$ if and only if $i=u$; and (b) $(i,j)\mathscr{L}(u,v)$ if and only if $k=s$ and $(j,v)\in\mathscr{H}(\in J_k)$.*

PROOF. (a) Let $(i,j)\mathscr{R}(u,v)$. Then, using (3), there exists $x,y\in I$ such that $i=u\circ x$ and $u=i\circ y$. Hence, $u\circ u=u\circ(u\circ x)=u\circ x=i$, and similarly $i\circ u=u$. Therefore, $i=u$. Conversely, let $i=u$. Since $j\beta_{(k,s)}\in J_s$, by (2), and J_s is a right group, there exists $b\in J_s$ such that $j\beta_{(k,s)}\otimes b=v$. Let $a\in I_k$. Therefore, using (2), (i,j) $(a,b)=(i,v)$. Similarly, there exists $c\in I_s$ and $d\in J_k$ such that $(i,v)(c,d)=(i,j)$. Therefore, $(i,j)\mathscr{R}(i,v)$.

(b) Suppose $k=s$ and $(j,v)\in\mathscr{H}(\in J_k)$. Then, there exists $x\in H_k$(the $\mathscr{H}$-class of J_k containing e_k such that $x\otimes j=v$. Let $b=x\beta_{(k,n)}\in J_n$. Hence, using the fact $(n,k)\to\beta_{(n,k)}$ is a homomorphism and (1), $b\beta_{(n,k)}=x\beta_{(k,n)}\beta_{(n,k)}=x\beta_{(k,k)}=x\otimes e_k=x$. Hence, $(u,b)(i,j)=(u,v)$. Similarly, there exists $c\in I_r$ such that $(i,c)(u,v)=(i,j)$. Conversely, suppose that $(i,j)\mathscr{L}(u,v)$. Thus there exists $x\in J_p$ with $p\geq k$ and $y\in J_q$ with $q\geq s$ such that $x\otimes j=v$ and $y\otimes v=j$. Therefore, $p=s$ and $q=k$. Thus, $s=k$. Thus, $x\otimes j=v$, with $x,j,v\in J_k$. Hence, since J_k is a right group, $(j,v)\in\mathscr{H}(\in J_k)$.

LEMMA 2.3. *(I,J,α,β) is a bisimple semigroup.*

PROOF. Let $(i,j),(u,v)\in(I,J,\alpha,\beta)$. Then, using lemma 2.2, $(i,j)\mathscr{R}(i,v)\mathscr{L}(u,v)$.

LEMMA 2.4. $E(I,J,\alpha,\beta)=\{(i,j):i\in I_n,\ j\in E(J_n),\ n\in N\}$.

PROOF. Let $(i,j)\in E(I,J,\alpha,\beta)$ and suppose $i\in I_n$ and $j\in J_k$. Using (3) and (2), $n=k$ and $j\beta_{(n,k)}\otimes j=j$. Hence, using (1), $j=j\beta_{(k,k)}\otimes j=j\otimes e_k\otimes j=j\otimes j$, and, thus, $j \in E(J_n)$. Conversely, let $i \in I_n$ and $j \in E(J_n)$. Thus, using (3), (2) and (1), $(i,j) \in E(I,J,\alpha,\beta)$.

LEMMA 2.5. *(I,J,α,β) is a regular bisimple semigroup.*

PROOF. Using lemma 2.4, $(i,e_n) \in E(I,J,\alpha,\beta)$. Hence, by a result of Clifford and Miller [1, theorem 2.11], and lemma 2.3, (I,J,α,β) is a regular bisimple semigroup.

LEMMA 2.6. *Let $T_n = \{(i,j) : i \in I_n$ and $j \in J\}$ and let $T = \bigcup(T_n : n \in N)$. Then, T is an ω-chain of the rectangular groups $(T_n : n \in N)$.*

PROOF. Let $(i,j), (u,v) \in T_n$. Using (1), $(i,j)(u,v) = (i, j \otimes v)$. Hence, using [1, theorem 1.27], T_n is a rectangular group. The last statement of the lemma is a consequence of (3) and (2).

LEMMA 2.7. *$E(T_n) = \{(i,j) : i \in I_n, j \in E(I_n)\}$. $E(T)$ is an ω-chain of the rectangular bands $(E(T_n) : nN)$.*

PROOF. The first statement of the lemma is an immediate consequence of lemmas 2.6 and 2.4. Let $(i,j), (u,v) \in E(T_n)$. Hence, $(i,j)(u,v) = (i,v)$. Thus, $E(T_n)$ is a rectangular band. Let $(i,j) \in E(T_n)$ and $(u,v) \in E(T_k)$. Using (3), (2), (1) and the fact that $E(J)$ is a semigroup, $(i,j)(u,v) \in E(T_{\max(n,k)})$. Therefore, $E(T)$ is an ω-chain of the rectangular bands $(E(T_n) : n \in N)$.

LEMMA 2.8. *T is the union of the maximal subgroups of (I,J,α,β).*

PROOF. Let X denote the union of the maximal subgroups of (I,J,α,β). Let $x \in X$. Hence, using lemmas 2.2 and 2.4, $x \mathscr{H} (i,j)$ for some $i \in I_n$ and $j \in E(J_n)$, say. Thus $x \in T_n$ by lemma 2.6. Let $x \in T$. Hence, since any rectangular group is a union of its subgroups, $x \in X$.

LEMMA 2.9. *$\mathscr{L}$ is a left congruence on $E(T)$.*

PROOF. Let X be any semigroup such that $E(X)$ is a semigroup and let $e, f \in E(X)$. Then, $(e,f) \in \mathscr{L}(\in X)$ if and only if $(e,f) \in \mathscr{L}(\in E(X))$. Let $(i,j), (u,v) \in E(I,J,\alpha,\beta)$. Thus, using lemma 2.4 and 2.2(b), $(i,j) \mathscr{L} (u,v)$ if and only if $j = v$. Hence, using (3), $\mathscr{L}$ is a left congruence on $E(T)$.

LEMMA 2.10. *$\mathscr{R}$ is a right congruence on T.*

PROOF. Using lemma 2.2 (a) and the multiplication on T_n given in the proof of lemma 2.6, $(i,j)\mathscr{R}(u,v)(\in T)$ if and only if $i=u$. Hence, using (3), $\mathscr{R}$ is a right congruence on T.

REMARK. Lemmas 2.2—2.5, 2.7, and 2.9 may also be obtained from the corresponging results of [3].

THEOREM 2.11. *(I,J,α,β) is a right generalized ω-$\mathscr{L}$-unipotent bisimple semigroup.*

PROOF. Combine lemmas 2.5—2.10.

THEOREM 2.12. *(I,J,α,β) is a right generalized ω-$\mathscr{L}$-unipotent bisimple semigroup, and conversely every such semigroup is isomorphic to some (I,J,α,β).*

PROOF. Combine theorems 2.11 and 1.13.

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