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THE EXTENDED SUM OF TWO RADICAL CLASSES

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In [6] Yu-Lee Lee and R.E. Propes defined the sum $\mathscr{A}+\mathscr{B}$ of two radical classes $\mathscr{A}$ and $\mathscr{B}$ in a universal class W of associative rings as follows:

$$\mathscr{A}+\mathscr{B}=\{R\in W:\mathscr{A}(R)+\mathscr{B}(R)=R\}.$$

They also found the necessary and sufficient conditions for this sum to be a radical class. The purpose of this paper is to extend this sum and to find conditions for which this extended sum will be a radical class.

We shall employ the following terms and notation throughout this paper:

W: a universal class of associative rings, i.e., a class of rings homomorphically closed and containing all ideals of all rings in W.

If $\mathscr{A}$ is a class of rings then any ring in $\mathscr{A}$ is called an *$\mathscr{A}$-ring.*

If $I\leq R$ (i.e., I is an ideal of a ring $R\in W$) then I is called an *$\mathscr{A}$-ideal* of R.

If $R\in W$ has an $\mathscr{A}$-ideal which contains all $\mathscr{A}$-ideals of R, it is called the *$\mathscr{A}$-radical* of R, denoted by $\mathscr{A}(R)$.

A ring $R\in W$ is *$\mathscr{A}$-semisimple* if R has no nonzero $\mathscr{A}$-ideals. The class of all $\mathscr{A}$-semisimple rings will be denoted by $S(\mathscr{A})$.

Recall [1] that a subclass $\mathscr{A}$ of a universal class W of rings is called a *radical class* if and only if the following conditions are satisfied:

(a) $\mathscr{A}$ is homomorphically closed,

(b) each ring $R\in W$ has an $\mathscr{A}$-radical $\mathscr{A}(R)$,

and (c) if $R\in W$ then $R/\mathscr{A}(R)\in S(\mathscr{A})$.

DEFINITION. Let $\mathscr{A}$ and $\mathscr{B}$ be two radical classes in W. For each $R\in W$ we set $(\mathscr{A}\oplus\mathscr{B})(R)=\sum I$ where $I\leq R$ and $I/(\mathscr{A}(R)+\mathscr{B}(R))\in\mathscr{A}\cup\mathscr{B}$.

DEFINITION. $\mathscr{A}\oplus\mathscr{B}=\{R\in W:\ (\mathscr{A}\oplus B)(R)=R\}$.

NOTE. $\mathscr{A}+\mathscr{B}\subset\mathscr{A}\oplus\mathscr{B}$ and $S(\mathscr{A}+\mathscr{B})\subset\{R\in W:(\mathscr{A}\oplus\mathscr{B})(R)=0\}$. (Recall [6] that $S(\mathscr{A}+\mathscr{B})=S(\mathscr{A})\cap S(\mathscr{B})$.)

PROPOSITION 1. *The class $\mathscr{A}\oplus\mathscr{B}$ is homomorphically closed.*

PROOF. Let $R\in\mathcal{A}\oplus\mathcal{B}$ and R/J be a homomorphic image of R. Now $R=\sum I$, where $I\leq R$ and $I/(\mathcal{A}(R)+\mathcal{B}(R))\in\mathcal{A}\cup\mathcal{B}$. Thus $R/J=(\sum I)/J$. Let $\mathcal{A}(R/J)=K/J$ and $\mathcal{B}(R/J)=L/J$. $\mathcal{A}(R/J)+\mathcal{B}(R/J)=K/J+L/J=(K+L)/J$. $(R/J)/\mathcal{A}(R/J)=(R/J)/(K/J)\cong R/K\ \in S(\mathcal{A})$ and $(R/J)/\mathcal{B}(R/J)=(R/J)/(L/J)\cong R/L\in S(\mathcal{B})$. Thus $\mathcal{A}(R)\subset K$ and $\mathcal{B}(R)\subset L$ and hence $\mathcal{A}(R)+\mathcal{B}(R)\subset K+L$. $(R/J)/(\mathcal{A}(R/J)+\mathcal{B}(R/J))=(R/J)/((K+L)/J)\cong R/(K+L)=(\sum I)/(K+L)$. $R/(\mathcal{A}(R)+\mathcal{B}(R))$ can be mapped homomorphically onto $(\sum I)/(K+L)$. By definition $I/(\mathcal{A}(R)+\mathcal{B}(R))\in\mathcal{A}\cup\mathcal{B}$. But $I/(\mathcal{A}(R)+\mathcal{B}(R))$ can be mapped homomorphically onto $I/(I\cap(K+L))\cong(I+K+L)/(K+L)$. Therefore $(I+K+L)/(K+L)\in\mathcal{A}\cup\mathcal{B}$. But $\sum_I(I+K+L)/(K+L)=((\sum_I I)+K+L)/(K+L)=R/(K+L)$. By definition, $(\mathcal{A}\oplus\mathcal{B})(R/J)=\sum(P/J)$, where $P\leq R$ and $P/(K+L)\in\mathcal{A}\cup\mathcal{B}$. Now $(I+K+L)/J\ \leq R/J$ and $(I+K+L)/(K+L)\in\mathcal{A}\cup\mathcal{B}$. Thus $\sum_I(I+K+L)/J\subset(\mathcal{A}\oplus\mathcal{B})(R/J)$. But $\sum_I(I+K+L)/J=((\sum_I I)+K+L)/J=R/J$.

Hence $(\mathcal{A}\oplus\mathcal{B})(R/J)=R/J$ i.e., $R/J\in\mathcal{A}\oplus\mathcal{B}$.

PROPOSITION 2. *$(\mathcal{A}\oplus\mathcal{B})(R)$ is an $(\mathcal{A}\oplus\mathcal{B})$-ideal of the ring R.*

PROOF. Set $P=(\mathcal{A}\oplus\mathcal{B})(R)$. Since $\mathcal{A}(R)+\mathcal{B}(R)\subset P$, we have $\mathcal{A}(R)\subset P$ and $\mathcal{B}(R)\subset P$. But $P\leq R$ so that $\mathcal{A}(R)=\mathcal{A}(P)$ and $\mathcal{B}(R)=\mathcal{B}(P)$. Let $I\leq R$ such that $I/(\mathcal{A}(R)+\mathcal{B}(R))\in\mathcal{A}\cup\mathcal{B}$. Then by definition $I\subset P$. Moreover, $I/(\mathcal{A}(R)+\mathcal{B}(R))=I/(\mathcal{A}(P)+\mathcal{B}(P))\leq P/(\mathcal{A}(P)+\mathcal{B}(P))$. Then, by definition, $I\subset(\mathcal{A}\oplus\mathcal{B})(P)$. Thus $(\mathcal{A}+\mathcal{B})(R)\subset(\mathcal{A}+\mathcal{B})(P)$, and hence $(\mathcal{A}\oplus\mathcal{B})(P)=(\mathcal{A}\oplus\mathcal{B})(R)=P$.

PROPOSITION 3. *Let $R\in W$. Then $(\mathcal{A}\oplus\mathcal{B})(R)$ is the largest $(\mathcal{A}\oplus\mathcal{B})$-ideal of R.*

PROOF. Let $I\leq R$ and let $I\in\mathcal{A}\oplus\mathcal{B}$. Then $I=(\mathcal{A}\oplus\mathcal{B})(I)=\sum J$, where $J\leq I$ and $J/(\mathcal{A}(I)+\mathcal{B}(I))\in\mathcal{A}\cup\mathcal{B}$. Without loss of generality assume $J/(\mathcal{A}(I)+\mathcal{B}(I))\in\mathcal{A}$. Then $J/(\mathcal{A}(I)+\mathcal{B}(I))\subset\mathcal{A}(I/(\mathcal{A}(I)+\mathcal{B}(I))$. But $I/(\mathcal{A}(I)+\mathcal{B}(I))\leq R/(\mathcal{A}(I)+\mathcal{B}(I))$ and $\mathcal{A}$ is a radical class. Thus, by Theorem 1 [2], $\mathcal{A}(I/(\mathcal{A}(I)+\mathcal{B}(I))\leq R/(\mathcal{A}(I)+\mathcal{B}(I))$. Set $\mathcal{A}(I/(\mathcal{A}(I)+\mathcal{B}(I)))=K/(\mathcal{A}(I)+\mathcal{B}(I))$. Then $K\leq R$ and $J\subset K$. Since $K/(\mathcal{A}(I)+\mathcal{B}(I))$ is in $\mathcal{A}$ and can be mapped homomorphically onto $K/(K\cap(\mathcal{A}(R)+\mathcal{B}(R)))\cong(K+\mathcal{A}(R)+\mathcal{B}(R)/(\mathcal{A}(R)+\mathcal{B}(R))$, we have $(K+\mathcal{A}(R)+\mathcal{B}(R))/(\mathcal{A}(R)+\mathcal{B}(R))\in\mathcal{A}\subset\mathcal{A}\cup\mathcal{B}$. Hence $K+\mathcal{A}(R)+\mathcal{B}(R)\subset(\mathcal{A}\oplus\mathcal{B})(R)$. Therefore $I=\sum J\subset(\mathcal{A}\oplus\mathcal{B})(R)$.

THEOREM 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be radical classes in a universal class W of rings. If $S(\mathcal{A})\subset\mathcal{B}$ or $S(\mathcal{B})\subset\mathcal{A}$, then $\mathcal{A}\oplus\mathcal{B}$ is a radical class.*

PROOF. It suffices to show that $R/(\mathcal{A}\oplus\mathcal{B})(R)\in S(\mathcal{A}+\mathcal{B})$. By definition,

$(\mathscr{A}\oplus\mathscr{B})(R/\mathscr{A}\oplus\mathscr{B})(R))=\Sigma I/(\mathscr{A}\oplus\mathscr{B})(R)$ where $I\leq R$ and $I/(J+K)\in\mathscr{A}\cup\mathscr{B}$ where $J/(\mathscr{A}\oplus\mathscr{B})(R)=\mathscr{A}(R/(\mathscr{A}\oplus\mathscr{B})(R))$ and $K/(\mathscr{A}\oplus\mathscr{B})(R)=\mathscr{B}(R/(\mathscr{A}\oplus\mathscr{B})(R))$. Without loss of generality, assume that $S(\mathscr{A})\subset\mathscr{B}$. Now, $(I/\mathscr{A}(R))/((\mathscr{A}(R)+\mathscr{B}(R))/\mathscr{A}(R))\in\mathscr{B}$, because $I/\mathscr{A}(R)\in S(\mathscr{A})\subset\mathscr{B}$. Since $I/(\mathscr{A}(R)+\mathscr{B}(R))\cong(I/\mathscr{A}(R))/((\mathscr{A}(R)+\mathscr{B}(R))/\mathscr{A}(R))$ we have $I/(\mathscr{A}(R)+\mathscr{B}(R))\in\mathscr{B}\subset\mathscr{A}\cup\mathscr{B}$. Hence $I\subset(\mathscr{A}\oplus\mathscr{B})(R)$, i.e., $(\mathscr{A}\oplus\mathscr{B})(R/(\mathscr{A}\oplus\mathscr{B})(R)=0$. Therefore $R/(\mathscr{A}\oplus\mathscr{B})(R)$ is $(\mathscr{A}\oplus\mathscr{B})$-semisimple.

EXAMPLE 1. Let M be the class of fields of two elements. UM denotes the upper radical determined by M. By [3], every UM-semisimple ring is a subdirect sum of M-rings. Moreover, subdirect sums of fields of two elements are Boolean rings [7]. Hence $R\in S(UM)$ which implies that R is a Boolean ring.

Let $\mathscr{R}$ be the class of all regular rings [R is a regular ring if for each $a\in R$, there exists an element $x\in R$ such that $axa=a$]. As proved in [5], $\mathscr{R}$ is a radical class. Moreover, every Boolean ring is regular. Therefore, all Boolean rings are contained in $\mathscr{R}$. Hence UM and $\mathscr{R}$ are two radical classes such that $S(UM)\subset\mathscr{R}$ and so satisfy the condition of the theorem.

THEOREM 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be two radical classes in W such that $S(\mathscr{A})\subset\mathscr{B}$, $S(\mathscr{B})\subset\mathscr{A}$ and $\mathscr{A}\cap\mathscr{B}=0$. Then $W=\mathscr{A}\oplus\mathscr{B}=\mathscr{A}+\mathscr{B}$.*

PROOF. Let $R\in W$. Each of $R/\mathscr{A}(R)$ and $R/\mathscr{B}(R)$ can be mapped homomorphically onto $R/(\mathscr{A}(R)+\mathscr{B}(R))$. Now $R/\mathscr{A}(R)\in S(\mathscr{A})\subset\mathscr{B}$ and $R/\mathscr{B}(R)\in S(\mathscr{B})\subset\mathscr{A}$. Therefore, $R/(\mathscr{A}(R)+\mathscr{B}(R))\in\mathscr{A}\cap\mathscr{B}=0$. Hence $R=\mathscr{A}(R)+\mathscr{B}(R)$ and so $R\in W$ implies $R\in\mathscr{A}+\mathscr{B}$. Hence $W=\mathscr{A}+\mathscr{B}$. But we have already noticed that $\mathscr{A}+\mathscr{B}\subset\mathscr{A}\oplus\mathscr{B}$. Hence $\mathscr{A}+\mathscr{B}=\mathscr{A}\oplus\mathscr{B}=W$.

EXAMPLE 2. Let W be the universal class of rings whose additive groups are p-primary for some prime p. Let p be a prime and set $T_p=\{R\in W\colon (R,+)$ is p-primary$\}$. Then T_p is a radical class [4]. Let $Q=\bigcup_{q\neq p} T$. We claim that Q is a radical class, in fact $Q=U(T_p)$, upper radical class determined by T_p where

$$UT_p=\{R\in W\colon\ R/I\notin T_p\ \ \forall I\lneqq R\}.$$

Now $R\in UT_p$ if and only if $R\cong R/(0)\in T_q$ for some $q\neq p$ if and only if $R\in Q$. Moreover, $ST_p\subset Q$, $SQ\subset T_p$ and $T_p\cap Q=0$. Hence $W=T_p\oplus Q=T_p+Q$.

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REFERENCES

[1] S. A. Amitsur, *Radicals in rings and bicategories*, Amer. J. Math. 76(1954), 100—125.

[2] T. Anderson, N. Divinski, A. Sulinski, *Hereditary radicals in associative and alternative rings*, Can. J. Math. 17(1965), 594—603.

[3] V. A. Andrunakievic, *Radicals of associative rings II*. Mat. Sb. (N. S.) 55(97) 1961, 329—346.

[4] E. P. Armendariz and W. G. Leavitt, *The hereditary property in the lower radical construction*, Can. J. Math (20) 1968, 474—476.

[5] T. L. Jenkins, *The theory of radicals and radical rings*, Ph. D. Thesis, University of Nebraska, 1966.

[6] Yu-Lee Lee and R. E. Propes, *The sum of two radical classes*, Kyungpook Math. Jour. vol 13(1), 81—86.

[7] H. M. McCoy, *Subdirect sums of rings*, Bull. Amer. Math. Soc. 53(1947), 856—877.