Kyungpook Math. J.
Volume 16, Number 1
June, 1976

INFINITESIMAL VARIATIONS OF ANTI-INVARIANT SUBMANIFOLDS OF A KAEHLERIAN MANIFOLD

By Kentaro Yano and Masahiro Kon

Introduction

Various authors (see, for example, [1], [7], [8], [9]) studied recently anti-invariant (or totally real) submanifolds of a Kaehlerian manifold.

On the other hand, one of the present authors [6] has studied infinitesimal variations of submanifolds applying the method developed in [3] and [4].

The main purpose of the present paper is to study infinitesimal variations which carry an anti-invariant submanifold into an anti-invariant submanifold. Such an infinitesimal variation will be called in this paper an anti-invariant variation.

In §1, we state formulas for anti-invariant submanifolds of a Kaehlerian manifold which we need later.

§2 is devoted to the study of infinitesimal variations which carry an anti-invaraiant submanifold into an anti-invariant submanifold. A necessary and sufficient condition for an infinitesimal variation to carry an anti-invariant submanifold into an anti-invariant submanifold is given by Theorem 2.1.

In §3, we consider what we call infinitesimal parallel variations and prove that a parallel variation is an anti-invariant variation.

In §4 and 5, we compute variations of f_b^x and f_y^x respectively and in §6, we study isometric variations.

The last §7 is devoted to the study of variations of the second fundamental tensors. In the later part of §7, we study anti-invariant normal variations which preserve f_b^x and mean curvature vector.

1. Anti-invariant submanifolds of a Kaehlerian manifold

Let M^{2m} be a real $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{U;x^h\}$ and with almost complex structure tensor $F_i^{\ h}$ and

Hermitian metric tensor g_{ji}, where, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{\bar{1}, \bar{2}, \cdots, \overline{2m}\}$. Then we have

(1.1) $$F_i{}^t F_t{}^h = -\delta_i^h, \qquad F_j{}^t F_i{}^s g_{ts} = g_{ji},$$

(1.2) $$\nabla_j F_i{}^h = 0,$$

where ∇_j denotes the operator of covariant differentiation with respect to the Christoffel symbols $\Gamma_{ji}{}^h$ formed with g_{ji}.

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and with metric tensor g_{cb}, where, here and in the sequel, the indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, n\}$. We assume that M^n is isometrically immersed in M^{2m} by the immersion $i: M^n \longrightarrow M^{2m}$ and identify $i(M^n)$ with M^n. We represent the immersion $i: M^n \longrightarrow M^{2m}$ locally by

(1.3) $$x^h = x^h(y^a)$$

and put

(1.4) $$B_b{}^h = \partial_b x^h, \quad (\partial_b = \partial/\partial y^b),$$

which are n linearly independent vectors of M^{2m} tangent to M^n.
Since the immersion i is isometric, we have

(1.5) $$g_{cb} = g_{ji} B_c{}^j B_b{}^i.$$

We denote by $C_y{}^h$ $2m-n$ mutually orthogonal unit normals to M^n, where, here and in the sequel, the indices $x, y, z, \cdots$ run over the range $\{n+1, n+2, \cdots, 2m\}$. Then the equations of Gauss are written as

(1.6) $$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h,$$

where ∇_c denotes the operator of van der Waerden-Bortolotti covariant differentiation along M^n and $h_{cb}{}^x$ are second fundamental tensors of M^n with respect to the normals $C_x{}^h$ and those of Weingarten as

(1.7) $$\nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h,$$

where

$$h_c{}^a{}_y = h_{cby} g^{ba} = h_{cb}{}^z g^{ba} g_{zy},$$

g^{ba} denoting covariant components of the metric tensor g_{cb} of M^n, and g_{zy} the metric tensor of the normal bundle.

If the transform by F of any vector tangent to M^n is always normal to M^n, that is, if there exists a tensor field $f_b^{\ x}$ of mixed type such that

$$F_i^{\ h} B_b^{\ i} = -f_b^{\ x} C_x^{\ h}, \tag{1.8}$$

we say that M^n is *anti-invariant* (or *totally real*) in M^{2m}.

For the transform by F of normal vectors $C_y^{\ h}$, we have equations of the form

$$F_i^{\ h} C_y^{\ i} = f_y^{\ a} B_a^{\ h} + f_y^{\ x} C_x^{\ h}, \tag{1.9}$$

where

$$f_y^{\ a} = f_b^{\ z} g^{ba} g_{zy}, \tag{1.10}$$

which can also be written as

$$f_{ya} = f_{ay}, \tag{1.11}$$

where $f_{ya} = f_y^{\ b} g_{ba}$ and $f_{ay} = f_a^{\ z} g_{zy}$.

From (1.8) and (1.9) we find (cf. [7], [9])

$$f_b^{\ y} f_y^{\ a} = \delta_b^a, \tag{1.12}$$

$$f_b^{\ y} f_y^{\ x} = 0, \tag{1.13}$$

$$f_y^{\ z} f_z^{\ a} = 0, \tag{1.14}$$

$$f_y^{\ z} f_z^{\ x} = -\delta_y^x + f_y^{\ a} f_a^{\ x}. \tag{1.15}$$

Equations (1.14) and (1.15) show that $f_y^{\ x}$ is an f-structure in the normal bundle of M^n if it does not vanish. Differentiating (1.8) and (1.9) covariantly along M^n, and using equations of Gauss and Weingarten, we find

$$h_{cb}^{\ \ x} f_x^{\ a} - h_c^{\ a}{}_x f_b^{\ x} = 0, \tag{1.16}$$

$$\nabla_c f_b^{\ x} = -h_{cb}^{\ \ y} f_y^{\ x}, \tag{1.17}$$

$$\nabla_c f_y^{\ a} = h_c^{\ a}{}_x f_y^{\ x}, \tag{1.18}$$

$$\nabla_c f_y^{\ x} = h_c^{\ a}{}_y f_a^{\ x} - h_{ca}^{\ \ x} f_y^{\ a}. \tag{1.19}$$

If $m=n$, from (1.12) we have $f_y^{\ a} f_a^{\ x} = \delta_y^x$ and consequently from (1.15) we find $f_y^{\ z} f_z^{\ x} = 0$, that is, $f_{zy} f^{zy} = 0$, $f_{zy} = f_z^{\ x} g_{xy}$ and $f^{zy} = f_x^{\ y} g^{xz}$ being skew-symmetric. Thus we have $f_y^{\ x} = 0$. In this case, equations (1.12)~(1.15) reduces to

$$f_b^{\ y} f_y^{\ a} = \delta_b^a, \quad f_b^{\ x} f_y^{\ b} = \delta_y^x. \tag{1.20}$$

2. Infinitesimal variations of anti-invariant submanifolds

We consider an infinitesimal variation of anti-invariant submanifold M^n of a Kaehlerian manifold M^{2m} given by

$$\bar{x}^h = x^h(y) + \xi^h(y)\varepsilon, \tag{2.1}$$

where $\xi^h(y)$ is a vector field of M^{2m} defined along M^n and ε is an infinitesimal. We then have

$$\bar{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon, \tag{2.2}$$

where $\bar{B}_b{}^h = \partial_b \bar{x}^h$ are n linearly independent vectors tangent to the varied submanifold. We displace $\bar{B}_b{}^h$ parallelly from the varied point $(\bar{x}^h)$ to the original point (x^h). We then obtain the vectors

$$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j \bar{B}_b{}^i \varepsilon$$

at the point (x^h), or

$$\tilde{B}_b{}^h = B_b{}^h + (\nabla_b \xi^h)\varepsilon \tag{2.3}$$

neglecting the terms of order higher than one with respect to ε, where

$$\nabla_b \xi^h = \partial_b \xi^h + \Gamma_{ji}{}^h B_b{}^j \xi^i. \tag{2.4}$$

In the sequel we always neglect terms of order higher than one with respect to ε. Thus putting

$$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h, \tag{2.5}$$

we have from (2.3)

$$\delta B_b{}^h = (\nabla_b \xi^h)\varepsilon. \tag{2.6}$$

Putting

$$\xi^h = \xi^a B_a{}^h + \xi^x C_x{}^h, \tag{2.7}$$

we have

$$\nabla_b \xi^h = (\nabla_b \xi^a - h_b{}^a{}_x \xi^x) B_a{}^h + (\nabla_b \xi^x + h_{ba}{}^x \xi^a) C_x{}^h. \tag{2.8}$$

Now we denote by $\bar{C}_y{}^h$ $2m-n$ mutually orthogonal unit normals to the varied submanifold and by $\tilde{C}_y{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by parallel displacement of $\bar{C}_y{}^h$ from the point $(\bar{x}^h)$ to (x^h). Then we have

$$\tilde{C}_y{}^h = \bar{C}_y{}^h + \Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j \bar{C}_y{}^i \varepsilon. \tag{2.9}$$

We put

$$\delta C_y^{\ h} = \dot{C}_y^{\ h} - C_y^{\ h} \tag{2.10}$$

and assume that $\delta C_y^{\ h}$ is of the form

$$\delta C_y^{\ h} = \eta_y^{\ h}\varepsilon = (\eta_y^{\ a} B_a^{\ h} + \eta_y^{\ x} C_x^{\ h})\varepsilon. \tag{2.11}$$

Then, from (2.9), (2.10) and (2.11), we have

$$\bar{C}_y^{\ h} = C_y^{\ h} - \Gamma_{ji}^{\ h}\xi^j C_y^{\ i}\varepsilon + (\eta_y^{\ a} B_a^{\ h} + \eta_y^{\ x} C_x^{\ h})\varepsilon. \tag{2.12}$$

Applying the operator δ to $B_b^{\ j} C_y^{\ i} g_{ji} = 0$ and using (2.6), (2.8), (2.11) and $\delta g_{ji} = 0$, we find

$$(\nabla_b \xi_y + h_{bay}\xi^a) + \eta_{yb} = 0,$$

where $\xi_y = \xi^z g_{zy}$ and $\eta_{yb} = \eta_y^{\ c} g_{cb}$, or

$$\eta_y^{\ a} = -(\nabla^a \xi_y + h_b^{\ a}{}_y \xi^b), \tag{2.13}$$

∇^a being defined to be $\nabla^a = g^{ac}\nabla_c$. Applying the operator δ to $C_y^{\ j} C_x^{\ i} g_{ji} = \delta_{yx}$ and using (2.11) and $\delta g_{ji} = 0$, we find

$$\eta_{yx} + \eta_{xy} = 0, \tag{2.14}$$

where $\eta_{yx} = \eta_y^{\ z} g_{zx}$.

We now assume that the infinitesimal variation (2.1) carries an anti-invariant submanifold into an anti-invariant submanifold, that is,

(2.15) $\quad F_i^{\ h}(x+\xi\varepsilon)\bar{B}_b^{\ i}$ are linear combinations of $\bar{C}_x^{\ h}$.

Now using $\nabla_j F_i^{\ h} = 0$ and (1.8), we see that

$$\begin{aligned} &F_i^{\ h}(x+\xi\varepsilon)\bar{B}_b^{\ i} \\ &= (F_i^{\ h} + \xi^j \partial_j F_i^{\ h}\varepsilon)(B_b^{\ i} + \partial_b \xi^i \varepsilon) \\ &= [F_i^{\ h} - \xi^j(\Gamma_{jt}^{\ h} F_i^{\ t} - \Gamma_{ji}^{\ t} F_t^{\ h})\varepsilon](B_b^{\ i} + \partial_b \xi^i \varepsilon) \\ &= F_i^{\ h} B_b^{\ i} + (F_i^{\ h}\nabla_b \xi^i + f_b^{\ x}\Gamma_{ji}^{\ h} C_x^{\ j}\xi^i)\varepsilon, \end{aligned}$$

that is, by (2.12),

$$\begin{aligned} &F_i^{\ h}(x+\xi\varepsilon)\bar{B}_b^{\ i} \\ &\qquad = -f_b^{\ x}\bar{C}_x^{\ h} + [F_i^{\ h}\nabla_b \xi^i + f_b^{\ y}(\eta_y^{\ a} B_a^{\ h} + \eta_y^{\ x} C_x^{\ h})]\varepsilon. \end{aligned} \tag{2.16}$$

Thus we see that (2.15) is equivalent to

(2.17) $F_i{}^h\nabla_b\xi^i+f_b{}^y\eta_y{}^aB_a{}^h$ are linear combinations of $C_x{}^h$.

On the other hand, using (2.8) and (2.13), we have

$$(2.18)\quad \begin{aligned} &F_i{}^h\nabla_b\xi^i+f_b{}^x\eta_x{}^aB_a{}^h \\ &=-(\nabla_b\xi^a-h_b{}^a{}_x\xi^x)f_a{}^yC_y{}^h+(\nabla_b\xi^y+h_{ba}{}^y\xi^a)(f_y{}^cB_c{}^h+f_y{}^xC_x{}^h) \\ &\quad -f_b{}^x(\nabla^a\xi_x+h_c{}^a{}_x\xi^c)B_a{}^h \\ &=[(\nabla_b\xi^x+h_{bc}{}^x\xi^c)f_x{}^a-f_b{}^x(\nabla^a\xi_x+h_c{}^a{}_x\xi^c)]B_a{}^h \\ &\quad +[(\nabla_b\xi^y+h_{ba}{}^y\xi^a)f_y{}^x-(\nabla_b\xi^a-h_b{}^a{}_y\xi^y)f_a{}^x]C_x{}^h. \end{aligned}$$

Thus (2.15) or (2.16) is equivalent to

$$(2.19)\quad (\nabla_b\xi^x+h_{bc}{}^x\xi^c)f_x{}^a=f_b{}^x(\nabla^a\xi_x+h_c{}^a{}_x\xi^c),$$

or, by (1.16), to

$$(2.20)\quad (\nabla_b\xi^x)f_x{}^a=f_b{}^x(\nabla^a\xi_x),$$

or, by (1.11), to

$$(2.21)\quad (\nabla_b\xi_x)f_a{}^x=(\nabla_a\xi_x)f_b{}^x.$$

Thus we have

THEOREM 2.1. *In order for an infinitesimal variation to carry an anti-invariant submanifold into an anti-invariant submanifold, it is necessary and sufficient that the variation vector* ξ^h *satisfies* (2.20) *or* (2.21).

COROLLARY 2.1. *If a vector field* ξ^h *defines an infinitesimal variation which carries an anti-invariant submanifold into an anti-invariant submanifold, then another vector field* ξ'^h *which has the same normal part as* ξ^h *has the same property.*

An infinitesimal variation given by (2.1) is called an *anti-invariant variation* if it carries an anti-invariant submanifold into an anti-invariant submanifold. For an infinitesimal variation given by (2.1), when $\xi^x=0$, that is, when the variation vector ξ^h is tangent to the submanifold we say that the variation is *tangential* and when $\xi^a=0$, that is, when the variation vector ξ^h is normal to the submanifold we say that the variation is *normal.*

Since $\nabla_c f_b{}^x$ is symmetric in c and b by (1.17), we see that (2.21) is equivalent to

(2.22) $$\nabla_b(\xi_x f_a^x)=\nabla_a(\xi_x f_b^x).$$

Thus we see

PROPOSITION 2.1. *If $\xi_x f_a^x$ is closed, then an infinitesimal variation is an anti-invariant variation.*

If $m>n$, then there exists a normal vector field ξ in the normal bundle such that $\xi_x f_a^x=0$. Therefore, from Proposition 2.1, we obtain

THEOREM 2.2. *If $m>n$, then there always exists an anti-invariant normal variation.*

The mean curvature vector H^h of M^n is given by $H^h=\frac{1}{n}g^{cb}\nabla_c B_b{}^h$. If C^h is a unit normal vector in the direction of H^h, then $H^h=\alpha C^h$ for some function α. We call α the mean curvature of M^n. If the second fundamental tensors of M^n is of the form $h_{ba}{}^x=g_{ba}H^x$, where $H^x=\frac{1}{n}g^{ba}h_{ba}x$, then M^n is sad to be totally umbilical.

Now we assume that M^n is totally umbilical and anti-invariant in M^{2m}, then (1.16) gives

(2.23) $$H^x f_x^a=0.$$

From (2.23) and Proposition 2.1, we have

THEOREM 2.3. *Let M^n be a not totally geodesic, totally umbilical, anti-invariant submanifold of a Kaehlerian manifold $M^{2m}(m>n)$. Then the normal variation defined by the mean curvature vector H^h carries M^n into an anti-invariant submanifold.*

If a tangent vector u^a satisfies

(2.24) $$\nabla_b u_a=\nabla_a u_b,$$

then an infinitesimal normal variation defined by $\xi^x=f^x{}_a u^a$ satisfies (2.22). Therefore we have

PROPOSITION 2.2. *If a tangent vector u^a satisfies (2.24), then the normal variation defined by $\xi^x=f^x{}_a u^a$ is anti-invariant.*

3. Parallel variation

Suppose that an infinitesimal variation $\bar{x}^h=x^h+\xi^h\varepsilon$ carries a submanifold $x^h=$

$x^h(y)$ into another submanifold $\bar{x}^h=\bar{x}^h(y)$ and the tangent space of the original submanifold at (x^h) and that of the varied submanifold at the corresponding point $(\bar{x}^h)$ are parallel. Then we say that the variation is *parallel.*

Since we have from (2.5), (2.6) and (2.8)

$$\bar{B}_b{}^h=[\delta_b{}^a+(\nabla_b\xi^a-h_b{}^a{}_x\xi^x)\varepsilon]B_a{}^h+(\nabla_b\xi^x+h_{ba}{}^x\xi^a)C_x{}^h\varepsilon, \tag{3.1}$$

we have

LEMMA 3.1 ([6]). *In order for an infinitesimal variation to be parallel, it is necessary and sufficient that*

$$\nabla_b\xi^x+h_{ba}{}^x\xi^a=0. \tag{3.2}$$

If (3.2) is satisfied, then (2.19) is satisfied. Thus we have

THEOREM 3.1. *A parallel variation is an anti-invariant variation.*

4. Variation of $f_b{}^x$

Suppose that an anti-invariant variation $\bar{x}^h=x^h+\xi^h\varepsilon$ carries an anti-invariant submanifold into an anti-invariant submanifold, that is, it is an anti-invariant variation. Then putting

$$F_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i=-(f_b{}^x+\delta f_b{}^x)\bar{C}_x{}^h, \tag{4.1}$$

we have, from (2.16), (2.18) and (2.19),

$$\begin{aligned}-(\delta f_b{}^x)\bar{C}_x{}^h=[(\nabla_b\xi^y+h_{ba}{}^y\xi^a)f_y{}^x\\ -(\nabla_b\xi^a-h_b{}^a{}_y\xi^y)f_a{}^x+f_b{}^y\eta_y{}^x]C_x{}^h\varepsilon,\end{aligned}$$

from which

$$\delta f_b{}^x=[(\nabla_b\xi^a-h_b{}^a{}_y\xi^y)f_a{}^x-(\nabla_b\xi^y+h_{ba}{}^y\xi^a)f_y{}^x-f_b{}^y\eta_y{}^x]\varepsilon. \tag{4.2}$$

Thus we have

PROPOSITION 4.1. *Suppose that an infinitesimal variation is anti-invariant. Then the variation of $f_b{}^x$ is given by* (4.2).

PROPOSITION 4.2. *An anti-invariant variation preserves $f_b{}^x$ if and only if*

$$(\nabla_b\xi^a-h_b{}^a{}_y\xi^y)f_a{}^x-(\nabla_b\xi^y+h_{ba}{}^y\xi^a)f_y{}^x-f_b{}^y\eta_y{}^x=0. \tag{4.3}$$

5. Variation of $f_y^{\ x}$

In this section we suppose that an infinitesimal variation $\bar{x}^h = x^h + \xi^h \varepsilon$ is anti-invariant. To find the variation of $f_y^{\ x}$, we apply the operator δ to

$$F_i^{\ h} C_y^{\ i} = f_y^{\ a} B_a^{\ h} + f_y^{\ x} C_x^{\ h}.$$

Then using $\delta F_i^{\ h} = 0$, (2.11) and (2.6), we find

$$\begin{aligned} &F_i^{\ h}(\eta_y^{\ a} B_a^{\ i} + \eta_y^{\ x} C_x^{\ i})\varepsilon \\ &\quad = (\delta f_y^{\ a}) B_a^{\ h} + f_y^{\ a} \nabla_a \xi^h \varepsilon + (\delta f_y^{\ x}) C_x^{\ h} \\ &\qquad + f_y^{\ z}(\eta_z^{\ a} B_a^{\ h} + \eta_z^{\ x} C_x^{\ h})\varepsilon, \end{aligned}$$

or

$$\begin{aligned} &[-\eta_y^{\ a} f_a^{\ x} C_x^{\ h} + \eta_y^{\ z}(f_z^{\ a} B_a^{\ h} + f_z^{\ x} C_x^{\ h})]\varepsilon \\ &\quad = (\delta f_y^{\ a}) B_a^{\ h} + f_y^{\ e}[(\nabla_e \xi^a - h_e^{\ a}{}_x \xi^x) B_a^{\ h} + (\nabla_e \xi^x + h_e^{\ x}{}_a \xi^a) C_x^{\ h}]\varepsilon \\ &\qquad + (\delta f_y^{\ x}) C_x^{\ h} + f_y^{\ z}(\eta_z^{\ a} B_a^{\ h} + \eta_z^{\ x} C_x^{\ h})\varepsilon, \end{aligned}$$

from which

$$\eta_y^{\ z} f_z^{\ a} \varepsilon = \delta f_y^{\ a} + f_y^{\ e}(\nabla_e \xi^a - h_e^{\ a}{}_x \xi^x)\varepsilon - f_y^{\ z}(\nabla^a \xi_z + h_b^{\ a}{}_z \xi^b)\varepsilon,$$

or, using (1.18)

(5.1) $$\delta f_y^{\ a} = [\xi^b \nabla_b f_y^{\ a} - f_y^{\ e} \nabla_e \xi^a + \eta_y^{\ x} f_x^{\ a} + f_y^{\ e} h_e^{\ a}{}_x \xi^x + f_y^{\ x} \nabla^a \xi_x]\varepsilon$$

and

$$[-\eta_y^{\ a} f_a^{\ x} + \eta_y^{\ z} f_z^{\ x}]\varepsilon = f_y^{\ e}(\nabla_e \xi^x + h_{ea}^{\ \ x} \xi^a)\varepsilon + \delta f_y^{\ x} + f_y^{\ z} \eta_z^{\ x}\varepsilon,$$

$$\delta f_y^{\ x} = [-f_y^{\ e}(\nabla_e \xi^x + h_{ea}^{\ \ x} \xi^a) + (\nabla^a \xi_y + h_c^{\ a}{}_y \xi^c) f_a^{\ x} + \eta_y^{\ z} f_z^{\ x} - f_y^{\ z} \eta_z^{\ x}]\varepsilon,$$

or, using (1.19),

(5.2) $$\delta f_y^{\ x} = [\xi^c \nabla_c f_y^{\ x} + \eta_y^{\ z} f_z^{\ x} - \eta_z^{\ x} f_y^{\ z} - f_y^{\ e}(\nabla_e \xi^x) + (\nabla^e \xi_y) f_e^{\ x}]\varepsilon,$$

or, using (2.13),

(5.3) $$\delta f_y^{\ x} = [\eta^x{}_e f_y^{\ e} - \eta_y^{\ a} f_a^{\ x} + \eta_y^{\ z} f_z^{\ x} - f_y^{\ z} \eta_z^{\ x}]\varepsilon.$$

Thus we have

PROPOSITION 5.1. *Suppose that an infinitesimal variation is anti-invariant. Then the variation of* $f_y^{\ x}$ *is given by* (5.2) *or* (5.3).

PROPOSITION 5.2. *An anti-invariant variation preserves the* f*-structure* $f_y^{\ x}$ *in the normal bundle if and only if*

$$\xi^c\nabla_c f_y^{\ x}+\eta_y^{\ z}f_z^{\ x}-\eta_z^{\ x}f_y^{\ z}-f_y^{\ e}(\nabla_e\xi^x)+(\nabla^e\xi_y)f_e^{\ x}=0, \tag{5.4}$$

or

$$\eta^x_{\ e}f_y^{\ e}-\eta_y^{\ e}f_e^{\ x}+\eta_y^{\ z}f_z^{\ x}-f_y^{\ z}\eta_z^{\ x}=0. \tag{5.5}$$

6. Isometric variations

First of all, applying the operator δ to (1.5) and using (2.6), (2.8) and $\delta g_{ji}=0$, we find (cf. [6])

$$\delta g_{cb}=(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)\varepsilon, \tag{6.1}$$

from which

$$\delta g^{ba}=-(\nabla^b\xi^a+\nabla^a\xi^b-2h^{ba}{}_x\xi^x)\varepsilon. \tag{6.2}$$

A variation of a submanifold for which $\delta g_{cb}=0$ is said to be *isometric.*

Now we assume that an anti-invariant variation preserves $f_b^{\ x}$, that is, $\delta f_b^{\ x}=0$. Then (1.12), (1.14) and (4.3) imply

$$\nabla_b\xi_c-h_{bcy}\xi^y=f_b^{\ y}f_c^{\ x}\eta_{yx}. \tag{6.3}$$

Thus, by (2.14), (6.1) and (6.3), we have $\delta g_{cb}=0$. Therefore we obtain

PROPOSITION 6.1. *If an anti-invariant variation preserves* $f_b^{\ x}$, *then the variation is isometric.*

We assume next that $m=n$ and the anti-invariant variation is normal. Then we have $f_y^{\ x}=0$ and hence (4.2) becomes

$$\delta f_b^{\ x}=-(h_b^{\ a}{}_y\xi^y f_a^{\ x}-f_b^{\ y}\eta_y^{\ x})\varepsilon. \tag{6.4}$$

If the variation moreover preserves $f_b^{\ x}$, then (6.1) and Proposition 6.1 show that $h_{cbx}\xi^x=0$. Thus (6.4) implies $f_b^{\ y}\eta_y^{\ x}=0$, from which $\eta_y^{\ x}=0$. Consequently

(2.11) reduces to

$$\delta C_y^{\ h}=\eta_y^{\ a}B_a^{\ h}\varepsilon. \tag{6.5}$$

PROPOSITION 6.2. *If $m=n$ and anti-invariant normal variation preserves $f_b^{\ x}$, then the variation of $C_y^{\ h}$ is given by* (6.5).

Furthermore, if the variation is parallel, then (2.13) gives $\eta_y^{\ a}=0$. Thus we have

PROPOSITION 6.3. *If $m=n$ and if a parallel anti-invariant normal variation preserves $f_b^{\ x}$, then it preserves $C_y^{\ h}$.*

7. Variations of the second fundamental tensors

In this section we compute infinitesimal variations of the second fundamental tensors (see [6]).

Suppose that v^h is a vector field of M^{2m} defined intrinsically along the submanifold M^n. When we displace the submanifold M^n by $\bar{x}^h=x^h+\xi^h(y)\varepsilon$ in the direction of ξ^h, we obtain a vector field $\bar{v}^h$ which is defined also intrinsically by the same rule along the varied submanifold. If we displace $\bar{v}^h$ back parallelly from the point $(\bar{x}^h)$ to (x^h), we get

$$\hat{v}^h=\bar{v}^h+\Gamma_{ji}^{\ \ h}(x+\xi\varepsilon)\xi^j\bar{v}^i\varepsilon$$

and hence, putting $\delta v^h=\hat{v}^h-v^h$, we find

$$\delta v^h=\bar{v}^h-v^h+\Gamma_{ji}^{\ \ h}\xi^j v^i\varepsilon.$$

Similarly we have

$$\delta\nabla_c v^h=\bar{\nabla}_c\bar{v}^h-\nabla_c v^h+\Gamma_{ji}^{\ \ h}\xi^j\nabla_c v^i\varepsilon,$$

that is,

$$\delta\nabla_c v^h=\nabla_c\bar{v}^h-\nabla_c v^h+(\partial_k\Gamma_{ji}^h+\Gamma_{kt}^h\Gamma_{ji}^t)\xi^k B_c^{\ j}v^i\varepsilon$$
$$+\Gamma_{ji}^{\ \ h}[(\partial_c\xi^j)v^i+\xi^j(\partial_c v^i)]\varepsilon.$$

On the other hand, we have

$$\nabla_c\delta v^h=\nabla_c\bar{v}^h-\nabla_c v^h+(\partial_j\Gamma_{ki}^{\ \ h}+\Gamma_{jt}^{\ \ h}\Gamma_{ki}^{\ \ t})\xi^k B_c^{\ j}v^i\varepsilon$$
$$+\Gamma_{ji}^{\ \ h}[(\partial_c\xi^j)v^i+\xi^j(\partial_c v^i)]\varepsilon.$$

From these equations we find

$$\delta\nabla_c v^h - \nabla_c \delta v^h = K_{kji}{}^h \xi^k B_c{}^j v^i \varepsilon,$$

where $K_{kji}{}^h$ is the curvature tensor of M^{2m}.

Similarly for a tensor field carrying three kinds of indices, say $T_{by}{}^h$, we have

$$\begin{aligned}(7.1)\qquad &\delta\nabla_c T_{by}{}^h - \nabla_c \delta T_{by}{}^h \\ &= K_{kji}{}^h \xi^k B_c{}^j T_{by}{}^i \varepsilon - (\delta\Gamma^a_{cb}) T_{ay}{}^h - (\delta\Gamma^x_{cy}) T_{bx}{}^h,\end{aligned}$$

$\delta\Gamma^a_{cb}$ and $\delta\Gamma^x_{cy}$ being the variation of the affine connection Γ^a_{cb} induced on M^n and that of the affine connection induced on the normal bundle of M^n respectivly. Applying formula (7.1) to $B_b{}^h$, we find

$$\delta\nabla_c B_b{}^h - \nabla_c \delta B_b{}^h = K_{kji}{}^h \xi^k B_c{}^j B_b{}^i \varepsilon - (\delta\Gamma^a_{cb}) B_a{}^h,$$

or using (1.6) and (2.6)

$$\delta(h_{cb}{}^x C_x{}^h) = (\nabla_c \nabla_b \xi^h + K_{kji}{}^h \xi^k B_c{}^j B_b{}^i)\varepsilon - (\delta\Gamma^a_{cb}) B_a{}^h,$$

from which, using (2.11),

$$\begin{aligned}&(\delta h_{cb}{}^x) C_x{}^h + h_{cb}{}^x (\eta_x{}^a B_a{}^h + \eta_x{}^y C_y{}^h)\varepsilon \\ &= (\nabla_c \nabla_b \xi^h + K_{kji}{}^h \xi^k B_c{}^j B_b{}^i)\varepsilon - (\delta\Gamma^a_{cb}) B_a{}^h.\end{aligned}$$

Thus we have

$$(7.2)\qquad \delta\Gamma^a_{cb} = (\nabla_c \nabla_b \xi^h + K_{kji}{}^h \xi^k B^{ji}_{cb}) B^a{}_h \varepsilon - h_{cb}{}^x \eta_x{}^a \varepsilon$$

and

$$(7.3)\qquad \delta h_{cb}{}^x = -h_{cb}{}^y \eta_y{}^x \varepsilon + (\nabla_c \nabla_b \xi^h + K_{kji}{}^h \xi^k B_c{}^j B_b{}^i) C^x{}_h \varepsilon,$$

from which

$$\begin{aligned}(7.4)\qquad \delta h_{cb}{}^x = {} & [\xi^d \nabla_d h_{cb}{}^x + h_{eb}{}^x (\nabla_c \xi^e) + h_{ce}{}^x (\nabla_b \xi^e) - h_{cb}{}^y \eta_y{}^x]\varepsilon \\ & + [\nabla_c \nabla_b \xi^x + K_{kji}{}^h C_y{}^k B^{ji}_{cb} C^x{}_h \xi^y - h_{ce}{}^x h_b{}^e{}_y \xi^y]\varepsilon.\end{aligned}$$

Since for a normal variation we have

$$\delta(g^{cb} h_{cb}{}^x) = 2h^{cb}{}_y \xi^y h_{cb}{}^x + g^{cb} \delta h_{cb} x,$$

we obtain from (7.4)

$$\begin{aligned}(7.5)\qquad \delta\left(\frac{1}{n} g^{cb} h_{cb}{}^x\right) = \frac{1}{n} [& g^{cb} \nabla_c \nabla_b \xi^x + K_{kji}{}^h C_y{}^k B^{ji} C^x{}_h \xi^y \\ & + h_{cb}{}^x h^{cb}{}_y \xi^y - h_a{}^{ay} \eta_y{}^x]\varepsilon,\end{aligned}$$

where $B^{ji} = B^{ji}_{cb} g^{cb}$.

In the sequel we suppose that $m=n$ and the anti-invariant variation preserves f_b^x. Since we have $h_{cby}\xi^y=0$ and $\eta_y^x=0$, (7.5) yields

PROPOSITION 7.1. *If $m=n$ and an anti-invariant normal variation preserves f_b^x, then we have*

$$\delta\left(\frac{1}{n}g^{cb}h_{cb}{}^x\right)=\frac{1}{n}[g^{cb}\nabla_c\nabla_b\xi^x+K_{kji}{}^hC_y{}^kB^{ji}C^x{}_h\xi^y]\varepsilon. \tag{7.6}$$

COROLLARY 7.1. *If $m=n$ and an anti-invariant normal variation preserves f_b^x, then it preserves the mean curvature vector if and only if*

$$g^{cb}\nabla_c\nabla_b\xi^x+K_{kji}{}^hC_y{}^kB^{ji}C^x{}_h\xi^y=0. \tag{7.7}$$

Substituting (7.7) into

$$\frac{1}{2}\Delta(\xi^x\xi_x)=\frac{1}{2}g^{cb}\nabla_c\nabla_b(\xi^x\xi_x)=(g^{cb}\nabla_c\nabla_b\xi^x)\xi_x+(\nabla_c\xi_x)(\nabla^c\xi^x),$$

we find

$$\frac{1}{2}\Delta(\xi^x\xi_x)=-K_{kjih}C_y{}^kB^{ji}C_x{}^h\xi^y\xi^x+(\nabla^c\xi^x)(\nabla_c\xi_x), \tag{7.8}$$

K_{kjih} being covariant components of the curvature tensor of M^{2m}.

If an anti-invariant submanifold M^n is compact and orientable, we find, from (7.8),

$$\int_M[(\nabla^c\xi^x)(\nabla_c\xi_x)-K_{kjih}C_y{}^kB^{ji}C_x{}^h\xi^y\xi^x]dV=0. \tag{7.9}$$

Thus we have

THEOREM 7.1. *Suppose that $m=n$ and an anti-invariant normal variation preserves f_b^x and the mean curvature vector. If M^n is compact and orientable and satisfies*

$$K_{kjih}C_y{}^kB^{ji}C_x{}^h\xi^y\xi^x\leqq 0,$$

then the variation is parallel.

Suppose that the ambient Kaehlerian manifold M^{2m} is of constant holomorphic sectional curvature k. Then we have

$$K_{kjih}=\frac{1}{4}k[g_{kh}g_{ji}-g_{jh}g_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih}]. \tag{7.10}$$

Suppose also that a submanifold M^m of M^{2m} is anti-invariant. Then we have

$$K_{kjih}C_y{}^kB^{ji}C_x{}^h=\frac{1}{4}(m+3)kg_{yx}. \tag{7.11}$$

Thus we have, from Theorem 7.1,

PROPOSITION 7.2. *Suppose that* M^{2m} *is a Kaehlerian manifold of constant holomorphic sectional curvature* $k\leqq 0$ *and that* M^m *is a compact orientable anti-invariant submanifold of* M^{2m}. *If an anti-invariant normal variation of* M^m *preserves* $f_b^{\ x}$ *and the mean curvature vector, then the variation is parallel and* $k=0$.

Suppose that the ambient Kaehlerian manifold has vanishing Bochner curvature tensor. Then we have (see [7])

$$(7.12)\qquad K_{kjih}=-[g_{kh}L_{ji}-g_{jh}L_{ki}+L_{kh}g_{ji}-L_{jh}g_{ki} +F_{kh}M_{ji}-F_{jh}M_{ki}+M_{kh}F_{ji}-M_{jh}F_{ki}-2(F_{kj}M_{ih}+M_{kj}F_{ih})],$$

where

$$L_{ji}=-\frac{1}{2(m+2)}K_{ji}+\frac{1}{8(m+1)(m+2)}Kg_{ji},$$

$$M_{ji}=-L_{jt}F_i^{\ t},$$

K_{ji} and K being the Ricci tensor and the scalar curvature of M^{2m} respectively.

Suppose also that a submanifold M^m of M^{2m} is anti-invariant. Then we have

$$(7.13)\qquad K_{kjih}C_y^{\ k}B^{ji}C_x^{\ h}=-[(m+3)L_{yx}+Lg_{yx}+3L_{cb}f_y^{\ c}f_x^{\ b}],$$

where

$$L_{yx}=L_{ji}C_y^{\ j}C_x^{\ i},\qquad L=L_{ji}B^{ji},\qquad L_{cb}=L_{ji}B_{cb}^{ji}.$$

But, on the other hand, we have

$$L_{cb}f_y^{\ c}f_x^{\ b}=L_{ji}B_c^{\ j}B_b^{\ i}f_y^{\ c}f_x^{\ b}=L_{ji}F_t^{\ j}C_y^{\ t}F_s^{\ i}C_x^{\ s}=L_{yx},$$

because of $L_{ji}F_t^{\ j}F_s^{\ i}=L_{ts}$. Thus we have from (7.13)

$$(7.14)\qquad K_{kjih}C_y^{\ k}B^{ji}C_x^{\ h}=-[(m+6)L_{yx}+Lg_{yx}].$$

Thus we have

PROPOSITION 7.3. *Suppose that* M^{2m} *is a Kaehlerian manifold with vanishing Bochner curvature tensor and that* M^m *is a compact orientable anti-invariant submanifold of* M^{2m}. *If an anti-invariant normal variation of* M^m *preserves* $f_b^{\ x}$ *and the mean curvature vector and*

$$[(m+6)L_{yx}+Lg_{yx}]\,\xi^y\xi^x\geqq 0,$$

then the variation is parallel.

Tokyo Institute of Technology,
Science University of Tokyo.

REFERENCES

[1] B. Y. Chen and K. Ogiue, *On totally real submanifolds*, Trans. Amer. Math. Soc., 193 (1974), 257—266.

[2] B. Y. Chen and K. Yano, *On the theory of normal variations*, to appear in J. Differential Geometry.

[3] J. A. Schouten, *Ricci-Calculus*, Springer-Verlag, (1954).

[4] K. Yano, *Sur la théorie des déformations infinitésimales*, J. of the Fac. of Sci. Univ. of Tokyo, 6 (1949), 1—75.

[5] K. Yano, *The theory of Lie derivatives and its applications*, North-Holland Publ. Co., Amsterdam, (1957).

[6] K. Yano, *Infinitesimal variations of submanifolds*, to appear in Kōdai Math. Sem. Rep.

[7] Y. Yano, *Differential geometry of totally real submanifolds*, to appear.

[8] K. Yano and M. Kon, *Totally real submanifolds of complex space forms*, to appear in Tòhoku Math. J.

[9] K. Yano and M. Kon, *Totally real submanifolds of complex space forms* II, to appear.