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A REMARK ON GOLDBACH'S CONJECTURE

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The notation here is the same as that found in [2], and this paper might well be considered an appendix to Chapter 3 of that text.

We assume $n>C_{11}$. For each n let E_n be those points in $[x_0, x_0+1]$ which are not in any closed neighborhood of radius x_0 about any rational number $\frac{h}{q}$ where $(h, q)=1$, $(q, n)=1$, and $q\leq\log^{15}n$.

It is a trivial consequence ([2] p.62) of the Prime Number Theorem that $\int_{E_n} f^2(x, n)\varepsilon(-nx)dx=o(n\log^{-1}n)$. In this paper I show that if this estimate could be improved to $o(n\log^{-\Delta}n)$ for some $\Delta>2$, then it would follow that every sufficiently large even integer can be expressed as the sum of two primes.

The result follows from a suitable modification of the construction found in Chapter 3 of [2]. Without any loss of generality we will assume that Δ is arbitrarily close to 2.

Let $r(n)$ be the number of representations of n as a sum of two primes. It is easy to see that

$$r(n)=\int_{x_0}^{x_0+1} f^2(x, n)\,\varepsilon(-nx)dx \quad \text{for any } x_0.$$

We decompose the above integral into

$$r(n)=\int_{E_n} f^2(x, n)\,\varepsilon(-nx)dx+\sum_{\substack{q\leq\log^{15}n\\(q,n)=1}}\ \sum_{\substack{0<h\leq q\\(h,q)=1}} T(h, q) \tag{100}$$

where

$$T(h, q)=\int_{\frac{h}{q}-x_0}^{\frac{h}{q}+x_0} f^2(x, n)\,\varepsilon(-nx)dx \tag{101}$$

It follows immediately from Theorem 58 in [2] and the trivial inequalities $|f(x, n)|\leq n$ and $|g(y, n)|\leq n$ and the fact that if $|a|\leq n$ and $|b|\leq n$, then $|a^2-b^2|\leq 2n|a-b|$ with $a=f\left(\frac{h}{q}+y,\ n\right)$ and $b=\frac{\mu(q)}{\phi(q)}g(y, n)$ that if $(h, q)=1$,

$|y|\leq x_0$, $q\leq\log^{15}n$, then

$$\left|f^2\left(\frac{h}{q}+y,n\right)-\frac{\mu^2(q)}{\phi^2(q)}g^2(y,n)\right|\leq 2n^2\log^{-69}n \tag{102}$$

By a change of variable $y=\left(x-\frac{h}{q}\right)$ we have

$$T(h,q)=\varepsilon\left(-\frac{nh}{q}\right)\int_{-x_0}^{x_0} f^2\left(\frac{h}{q}+y,\ n\right)\varepsilon(-ny)dy \tag{103}$$

However,

$$\left|\varepsilon\left(-\frac{nh}{q}\right)\int_{-x_0}^{x_0} f^2\left(\frac{h}{q}+y,\ n\right)\varepsilon(-ny)dy-\frac{\mu^2(q)}{\phi^2(q)}\varepsilon\left(-\frac{nh}{q}\right)\int_{-x_0}^{x_0} g^2(y,n)\,\varepsilon(-ny)dy\right|$$

$$\leq\int_{-x_0}^{x_0}\left|f^2\left(\frac{h}{q}+y,n\right)-\frac{\mu^2(q)}{\phi^2(q)}g^2(y,n)\right|dy\leq 2\int_{-x_0}^{x_0} n^2\log^{-69}n\,dy$$

$$=4x_0n^2\log^{-69}n=4n\log^{-54}n.$$

Now let

$$T_1(n)=\int_{-x_0}^{x_0} g^2(y,n)\,\varepsilon(-ny)dy:$$

so that by (103) and the above we have that if $(h,q)=1$ and $q\leq\log^{15}n$

$$\left|T(h,q)-\frac{\mu^2(q)}{\phi^2(q)}T_1(n)\,\varepsilon\left(-\frac{nh}{q}\right)\right|\leq 4n\log^{-54}n \tag{104}$$

Let

$$T(n)=\sum_{m_1,m_2}\log^{-1}m_1\log^{-1}m_2 \tag{105}$$

With the condition of summation $m_1\geq 2$, $m_2\geq 2$, and $m_1+m_2=n$.
It is easy to see that

$$T(n)=\int_{-\frac{1}{2}}^{\frac{1}{2}} g^2(y,n)\,\varepsilon(-ny)dy \tag{106}$$

Also, it is clear that the number of terms on the right-hand side of (105) is $(n-3)$, and each term is greater than $\log^{-2}n$ and less than 1; so that

$$\frac{1}{3}n\log^{-2}n<T(n)<n \tag{107}$$

Now

$$\left|\sum_{m=2}^{m_1} \varepsilon(my)\right| \leq \frac{1}{|\sin \pi y|} \leq \frac{1}{2|y|};\ \left(m_1 \geq 2, 0<|y|\leq \frac{1}{2}\right).$$

Hence by definition of $g(y, n)$ and Abel's lemma,

$$|g(y, n)| < |y|^{-1} \qquad \left(0<|y|\leq \frac{1}{2}\right);$$

so that

$$\left|T(n)-T_1(n)\right| \leq 2\int_{x_0}^{\frac{1}{2}} y^{-2}dy = 2x_0^{-1} = 2n\ \log^{-15} n \tag{108}$$

Hence, for $(h, q)=1$, $q\leq \log^{15} n$

$$\left|\varepsilon\left(-\frac{nh}{q}\right)\right|\left|\frac{\mu^2(q)}{\phi^2(q)}\right|\left|T(n)-T_1(n)\right| \leq \frac{1}{\phi^2(q)}(2n\ \log^{-15} n),$$

and combining this fact with (104) we have: (109)

For $(h, q)=1$, $q\leq \log^{15} n$

$$\left|T(h, q)-\frac{\mu^2(q)}{\phi^2(q)}T(n)\,\varepsilon\left(-\frac{nh}{q}\right)\right| \leq 4n\ \log^{-54} n + \frac{1}{\phi^2(q)}(2n\ \log^{-15} n);$$

so that adding (109) $\phi(q)$ times for some fixed $q\leq \log^{15} n$ we have: (110)

$$\left|\sum_{\substack{0<h\leq q\\(h,q)=1}} T(h, q) - \frac{\mu^2(q)}{\phi^2(q)}T(n)\sum_{\substack{0<h\leq q\\(h,q)=1}} \varepsilon\left(-\frac{nh}{q}\right)\right|$$

$$\leq (4n\ \log^{-54} n)\phi(q) + \frac{1}{\phi^{4/3}(q)}(2n\ \log^{-15} n)\ \phi^{1/3}(q).$$

But $\phi(q)\leq \log^{15} n$ and ([4] p. 55)

$$\sum_{\substack{0<h\leq q\\(h,q)=1}} \varepsilon\left(-\frac{nh}{q}\right) = C_q(n);$$

so that it follows immediately from (110) that: (111)

$$\left|\sum_{\substack{0<h\leq q\\(h,q)=1}} T(h, q) - \frac{\mu^2(q)}{\phi^2(q)}\ T(n)C_q(n)\right| \leq 4n\ \log^{-39} n + \frac{1}{\phi^{4/3}(q)}(2n\ \log^{-10} n).$$

Considering only those $q\leq \log^{15} n$ such that $(q, n)=1$ we have: (112)

$$\left|\sum_{\substack{q\leq \log^{15} n\\(q,n)=1}}\ \sum_{\substack{0<h\leq q\\(h,q)=1}} T(h, q) - T(n)\sum_{\substack{q\leq \log^{15} n\\(q,n)=1}} \frac{\mu^2(q)}{\phi^2(q)}\,C_q(n)\right|$$

$$\leq (4n\ \log^{-39} n)(\log^{15} n) + \left[\sum_{q\leq \log^{15} n}\frac{1}{\phi^{4/3}(q)}\right](2n\ \log^{-10} n)$$

$$\leq 4n \log^{-24} n + C_1(2n \log^{-10} n) \leq C_2\, n \log^{-10} n;$$

since by Theorem 327 in [4]

$$\sum_{q \leq \log^{15} n} \frac{1}{\phi^{4/3}(q)} \leq C_1 \quad (C_1 \text{ independent of } n).$$

Hence combining (100), (112) and the unproved statement:

$$\left| \int_{E_n} f^2(x, n)\, \varepsilon(-nx)\, dx \right| \leq C_3\, n \log^{-\Delta} n \text{ for some } \Delta > 2$$

we have

$$\left| r(n) - T(n) \sum_{\substack{q \leq \log^{15} n \\ (q,n)=1}} \frac{\mu^2(q)}{\phi^2(q)} C_q(n) \right| \leq C_4\, n \log^{-\Delta} n \tag{113}$$

Let

$$S(n) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\phi^2(q)} C_q(n) D_q(n)$$

where

$$D_q(n) = \begin{cases} 1 & \text{if } (q, n) = 1 \\ 0 & \text{if } (q, n) > 1 \end{cases}$$

$$\left| S(n) - \sum_{\substack{q \leq \log^{15} n \\ (q,n)=1}} \frac{\mu^2(q)}{\phi^2(q)} C_q(n) \right| = \left| \sum_{q > \log^{15} n} \frac{\mu^2(q)}{\phi^2(q)} C_q(n) D_q(n) \right|$$

$$\leq \sum_{\substack{q > \log^{15} n \\ q \text{ square free}}} \frac{1}{\phi^2(q)};$$

since $\mu^2(q) = 0$ if q is not square free, and by Theorem 272 in [4] if q is square free and $(q, n) = 1$, then $|C_q(n)| = 1$. Hence

$$\left| S(n) - \sum_{\substack{q \leq \log^{15} n \\ (q,n)=1}} \frac{\mu^2(q)}{\phi^2(q)} C_q(n) \right| \leq C_5 \log^{-14} n,$$

by Theorem 327 in [4].

Combining this fact with (107) and (113) we have

$$\left| S(n)T(n) - T(n) \sum_{\substack{q \leq \log^{15} n \\ (q,n)=1}} \frac{\mu^2(q)}{\phi^2(q)} C_q(n) \right| \leq C_5\, n \log^{-14} n. \tag{114}$$

Combining (113) and (114) we have

$$|r(n) - S(n)T(n)| \leq C_6\, n \log^{-\Delta} n \tag{115}$$

Let

$$f(q)=\frac{\mu^2(q)}{\phi^2(q)}C_q(n)D_q(n)$$

By Theorem 60, Theorem 67, and Theorem 262 in [4] f is a multiplicative function of q. Also,

$$\sum_{q=1}^{\infty}|f(q)|\le n\sum_{q=1}^{\infty}\frac{1}{\phi^2(q)}<\infty \quad \text{for each } n;$$

so that by Theorem 2 in [2] we have for each n:

$$S(n)=\prod_p\sum_{m=0}^{\infty}f(p^m).$$

But

If $m=0$, $f(p^m)=f(p^0)=f(1)=\frac{\mu^2(1)}{\phi^2(1)}C_1(n)D_1(n)=1.$

if $m=1$, $f(p^1)=f(p)=\frac{\mu^2(p)}{\phi^2(p)}C_p(n)D_p(n)=\frac{C_p(n)D_p(n)}{(p-1)^2};$

If $m\ge 2$, $\mu(p^m)=0$; so that $f(p^m)=0$; so that

$$S(n)=\prod_p\left(1+\frac{C_p(n)D_p(n)}{(p-1)^2}\right).$$

Clearly, if n is even, $D_2(n)=0$; so that since $C_p(n)=(p-1)$ if $(p, n)>1$ and $C_p(n)=-1$ if $(p, n)=1$.

$$S(n)=\prod_{p>2}\left(1+\frac{C_p(n)D_p(n)}{(p-1)^2}\right)\ge\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$$

$$\ge\prod_{m=2}^{\infty}\left(1-\frac{1}{m^2}\right)=\frac{1}{2}.$$

Combining this fact with (107) and (115) it follows that every sufficiently large even integer can be expressed as the sum of two primes.

REMARK. Let $x_0^*=x_0n^{-\varepsilon}$ where $0<\varepsilon<1$. Let E_n^* be those points in $[x_0, x_0+1]$ which are not in any closed neighborhood of radius x_0^* about any rational number $\frac{h}{q}$ where $(h, q)=1$ and $q\le\log^{15}n$. Clearly

$$E_n\subset(E_n^*\cup E_n^{**})$$

where

$$E_n^{**}=\bigcup_{\substack{(h,q)=1\\(q,n)>1\\q\le\log^{15}n\\0<h\le q}}\left[\frac{h}{q}-x_0^*,\quad \frac{h}{q}+x_0^*\right].$$

But

$$E_n^* \cap E_n^{**} = \phi$$

and

$$\int_{E_n^{**}} |f^2(x, n)| dx \leq 2x_0^* n^2 \log^{30} n \leq C_7(n \log^{-3} n);$$

so that if

$$\int_{E_n^*} |f^2(x, n)| dx = o\left(\frac{n}{\log^{\Delta} n}\right) \text{ for some } \Delta > 2, \tag{116}$$

then

$$\int_{E_n} f^2(x, n)\, \varepsilon\, (-nx) dx = o\left(\frac{n}{\log^{\Delta} n}\right) \text{ for some } \Delta > 2.$$

Theorem 56 on page 54 in [2] states that if (153): $n \log^{-3} n < v \leq n$, (154): $\log^{15} n < q \leq n \log^{-15} n$, (155): $(h, q) = 1$, then (156): $\left| f\left(\frac{h}{q}, v\right)\right| = o(n \log^{-3} n)$. Fix $\varepsilon > 0$, arbitrarily small. Consider (153)*: $n^{1/2} \log^{-(1+\varepsilon)} n < v \leq n$; (154)*: $\log^{15} n < q \leq n^{1+\varepsilon} \log^{-15} n$; (156)*: $\left| f\left(\frac{h}{q}, v\right)\right| = o(n^{1/2} \log^{-(1+\varepsilon)} n)$. It is easy to see (cf. [2] p.62) that if it could be shown that (153)*, (154)* and (155) imply (156)*, then (116) would follow.

All my results may be known to others.

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