Kyungpook Math. J.
Volume 16, Number 1
June, 1976

ANOTHER APPROACH TO SOME RECURSION THEOREMS OF LANDAU

By C. J. Mozzochi

In the Preface for the Teacher in [2] Landau discusses some logical difficulties in Peano's original definitions of addition, multiplication and generalized sums and products of natural numbers (cf. [4] for a more modern viewpoint). In his pursuit of the elimination of these difficulties he employs throughout [2] proofs which although very elementary are nevertheless rather subtle. In this note I outline a method of combining portions of Landau's original proofs with a method of proof found in Section 12 of [1] in such a way that: (A) The set of natural numbers can still be taken as a set of undefined objects together with the (unproved) Peano Axioms. (B) Although the resulting proofs are as elementary as Landau's original proofs, all of the subtleties found in his original proofs appear, to me at least, to have been eliminated.

In [3] I have completely reconstructed [2]. However, in this note I modify only the proofs of the recursion theorems found in [2]; so that the rest of [2] remains undisturbed.

The reader who would prefer a rigorous, axiomatic treatment in which the natural numbers are constructed from sets and in which the Peano Axioms are derived as theorems of set theory is referred to [5].

In the sequel all the theorems are numbered as in [2]; N denotes the set of natural numbers and E denotes the set of complex numbers.

I now suggest the following modifications be made in [2]:

I. Immediately before Theorem 4 prove the following.

THEOREM A. *Let A be an arbitrary set and let c be an arbitrary element in A and let f be an arbitrary function from A into A. Then there exists a unique function g from N into A such that $g(1)=c$ and $g(a')=f(g(a))$ for all $a\in N$.*

PROOF. (Existence)

Let $C=\{B\mid B\subset N\times A,\ (1,\ c)\in B;\ (a,\ x)\in B$ implies $(a',\ f(x))\in B\}$. Clearly $(N\times A)\in C$; so that $C\neq\phi$. Let $g=\bigcap_{B\in C} B$. Clearly, $g\in C$. To show that g is a function defined on N we have to show that for each $a\in N$ $(a,x)\in g$ and $(a,y)\in g$ implies $x=y$. Let $M=\{a\in N\mid(a,x)\in g$ and $(a,y)\in g$ implies $x=y\}$. Suppose $1\notin M$. Then $(1,d)\in g$ and $d\neq c$. Consider $h=g-\{(1,d)\}$. $(1,c)\in h$ and if $(a,x)\in h$, then $(a',f(x))\in h$; since $a'\neq 1$. Hence $h\in C$ which is not possible. Hence $1\in M$. Suppose $a\in M$. Then there exists one and only one element $x\in A$ such that $(a,x)\in g$; so that $(a',f(x))\in g$. Suppose $a'\notin M$. Then $(a',z)\in g$ and $z\neq f(x)$. Consider $h=g-\{(a',z)\}$. $(1,c)\in h$ and $a'\neq 1$. Suppose $(b,t)\in h$ and consider $(b',f(t))$.

Case 1. $b=a$. Then $t=x$ and since $f(x)\neq z$, $(b',f(t))\neq(a',z)$; so that $(b',f(t))\in h$.

Case 2. $b\neq a$. Then $b'\neq a'$; so that $(b',f(t))\neq(a',z)$; so that $(b',f(t))\in h$. Hence $h\in C$ which is not possible. Hence $a'\in M$ and $M=N$.

(Uniqueness)

Suppose there exists a function h from N into A such that $h(1)=c$ and $h(a')=f(h(a))$ for all $a\in N$. Let $M=\{a\in N\mid g(a)=h(a)\}$. $g(1)=h(1)=c$; so that $1\in M$. Suppose $a\in M$. Then $g(a)=h(a)$; so that $g(a')=f(g(a))=f(h(a))=h(a')$; so that $a'\in M$.

II. Prove Theorem 4 in the following way.

PROOF. Fix $a\in N$. By Theorem A with $A=N, c=a'$ and f defined $f(x)=x'$ for each $x\in N$ there exists a unique function, f_a, from N into N such that

I. $f_a(1)=a'$

II. $f_a(b')=f(f_a(b))=(f_a(b))'$ for each $b\in N$.

III. Immediately after Theorem 4 prove the following.

THEOREM B. *For $a=1$ let T_1 be a function from N into N defined $T_1(b)=b'$ for each $b\in N$.*

For $a\in N$ let $T_{a'}$ be a function from N into N defined $T_{a'}(b)=T_1(f_a(b))=(f_a(b))'$ for each $b\in N$.

Then $T_1=f_1$ and $T_{a'}=f_{a'}$ for each $a\in N$ where f_a is that of Theorem 4.

PROOF. This follows from the uniqueness of f_a for each $a\in N$ and the following calculation:

$T_1(1)=1'$ and $T_1(b')=(b')'=(T_1(b))'$ for each $b\in N$. Also, for each $a\in N$, $T_{a'}(1)=(f_a(1))'=(a')'$ and $T_{a'}(b')=(f_a(b'))'=((f_a(b))')'=(T_{a'}(b))'$ for each $b\in N$.

IV. Prove Theorem 28 in the following way:

PROOF. Fix $a \in N$. Let F be a function from N into N defined $F(x)=(x+a)$ for all $x \in N$. Then by Theorem A with $A=N$, $c=a$, and $f=F$ there exists a unique function, f_a, from N into N such that

I. $f_a(1)=a$

II. $f_a(b')=F(f_a(b))=f_a(b)+a$ for each $b \in N$.

V. Immediately after Theorem 28 prove the following:

THEOREM C. *For $a=1$ let T_1 be a function from N into N defined $T_1(b)=b$ for each $b \in N$.*

For $a \in N$ let $T_{a'}$ be a function from N into N defined $T_{a'}(b)=T_1(f_a(b))+b=f_a(b)+b$ for each $b \in N$. Then $T_1=f_1$ and $T_{a'}=f_{a'}$ for each $a \in N$ where f_a is that of Theorem 28.

PROOF. This follows from the uniqueness of f_a for each $a \in N$ and the following calculation:

$T_1(1)=1$ and $T_1(b')=b'=b+1=T_1(b)+1$ for each $b \in N$. Also, for each $a \in N$, $T_{a'}(1)=f_a(1)+1=a+1=a'$ and $T_{a'}(b')=f_a(b')+b'=(f_a(b)+a)+b'=f_a(b)+(a+b')=f_a(b)+(a+b)'=f_a(b)+(a'+b)=f_a(b)+(b+a')=(f_a(b)+b)+a'=T_{a'}(b)+a'$.

VI. Immediately before Theorem 275 prove the following:

THEOREM D. *Let f be an arbitrary function from I_p, the set of integers greater than zero, into E. Then there exists a unique function g from I_p into E such that $g(1)=f(1)$ and $g(a+1)=g(a)+f(a+1)$ for all $a \in I_p$.*

PROOF. Let

$C=\{B \mid B \subset I_p \times E,\ (1, f(1)) \in B;\ (a, x) \in B \text{ implies } (a+1,\ x+f(a+1)) \in B\}$.

Clearly, $(I_p \times E) \in C$; so that $C \neq \phi$. Let $g=\bigcap_{B \in C} B$.

Now proceed exactly as in the proof of Theorem A.

VII. Then prove

THEOREM E. *Let f be an arbitrary function from I_p into E. Then there exists a unique function g from I_p into E such that $g(1)=f(1)$ and $g(a+1)=g(a) \cdot f(a+1)$ for all $a \in I_p$.*

PROOF. Replace $+$ with $\cdot$ in the proof of Theorem D.

Ⅷ. Then prove

THEOREM F. *Let f be a function from* $[1, x]=\{a\in I_p | 1\leq a\leq x\}$ *into* E. *Then there exists a unique function* $g_{x,f}$ *from* $[1, x]$ *into* E *such that* $g_{x,f}(1)=f(1)$ *and* $g_{x,f}(a+1)=g_{x,f}(a)+f(a+1)$ *for* $a<x$.

PROOF. This is immediate by Theorem D by defining $f(a)=f(x)$ for each $a\geq x$ and letting $g_{x,f}$ be the restriction of g to $[1, x]$.

Ⅸ. Then prove

THEOREM G. *Let f be a function from* $[1, x]$ *into* E. *Then there exists a unique function* $g_{x,f}$ *from* $[1, x]$ *into* E *such that* $g_{x,f}(1)=f(1)$ *and* $g_{x,f}(a+1)=g_{x,f}(a)\cdot f(a+1)$ *for* $a<x$.

PROOF. This is immediate by Theorem E by defining $f(a)=f(x)$ for each $a\geq x$ and letting $g_{x,f}$ be the restriction of g to $[1, x]$.

Ⅹ. Now let $*$ signify either $+$ or $\cdot$ then prove

THEOREM H. *Let f be a function from* $[1, x+1]$ *into* E. *Let T be a function from* $[1, x+1]$ *into* E *defined*: $T(a)=g_{x,f}(a)$ *for* $1\leq a\leq x$ *and let* $T(x+1)=g_{x,f}(x)*f(x+1)$. *Then* $T(a)=g_{x+1,f}(a)$ *for each* $a\in[1, x+1]$.

PROOF. This follows from the uniqueness of $g_{x+1,f}$ and the following calculation: $T(1)=g_{x,f}(1)=f(1)$. Suppose $a<x$; so that $a+1\leq x$. Then $T(a+1)=g_{x,f}(a+1)=g_{x,f}(a)*f(a+1)=T(a)*f(a+1)$. Suppose $a=x$. Then $T(a+1)=g_{x,f}(a)*f(a+1)=T(a)*f(a+1)$; so that $T(a+1)=T(a)*f(a+1)$ for $a<x+1$.

Ⅺ. Now proceed directly to Theorem 277.

Box 1315
Hartford, Conn. 06101
U.S.A.

REFERENCES

[1] Halmos, P.R., *Naive Set Theory,* New York, 1960.

[2] Landau, E., *Grundlagen der Analysis*, Leipzig, 1930. Reprinted, New York, 1946.

[3] Mozzochi, C.J., *Foundations of Analysis,* Landau Revisited (publication pending).

[4] Robinson, J., *Definability and decision problems in arithmetic,* Journal of Symbolic Logic, Vol. 14 (1949), pp. 98—114.

[5] Suppes, P., *Axiomatic Set Theory,* New York, 1960. Reprinted, New York, 1972.