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# PARTIALLY ORDERED SETS AND GROUPOIDS

By J. Neggers

Let $(S, \leq)$ and $(T, \leq)$ be partially ordered sets (posets). Then a function $f: S \rightarrow T$ is *regular order preserving* if it is order preserving and if the image of any two incomparable elements is either a singleton or a pair of incomparable elements. The composition of regular order preserving functions is a regular order preserving function. Hence the collection of posets and regular order preserving functions is a category.

Now, let $S$ be a groupoid, i.e., a set with a binary operation, denoted in the usual fashion $(x,y) \rightarrow xy$. Suppose that this binary operation satisfies the following conditions:

1. $xy \in \{x,y\}$ for all $x,y \in S$
2. $x(yx)=yx$ for all $x,y \in S$
3. $(xy)(yz)=(xy)z$ for all $x,y,z \in S$.

We shall call such groupoids *pogroupoids*.

As usual, if $S$ and $T$ are groupoids, then a homomorphism from $S$ to $T$ is a mapping $f$: $S \rightarrow T$ such that $f(xy)=f(x)f(y)$. It follows that the class of pogroupoids and homomorphism is a category.

In this paper we prove the following theorem:

THEOREM. *There is a natural (functorial) isomorphism between the category of pogroupoids (and homomorphisms) and the category of posets (and regular order preserving functions).*

The advantage of the theorem is that we may in this way consider posets as algebraic objects, viz., pogroupoids and so presumably describe some properties of posets as algebraic properties of pogroupoids. Thus $(S, \leq)$ is a totally ordered set if and only if its associated pogroupoid is commutative. Similarly if $(S, \leq)$ is a poset such that incomparability is transitive, then its associated pogroupoid is associative and conversely. We use this characterization to show that if $S$ is an associative pogroupoid then $(S, \leq)$ is an ordinal sum of posets $(S_i, \leq)$ where $(S_i, \leq)$ is a set $S_i$ along with the diagonal relation. Also, if $(S, \leq)$ is of the latter type, then its associated pogroupoid is associative. As another example we show that if $S$ is a pogroupoid such that $x(yz)=y(xz)$ for all $x, y, z \in S$, then $S$ is a poset such that the set of upper bounds of a subset $A$ of $S$ in $(S, \leq)$ is a chain or the empty set. A method for generating results is discussed.

If we consider those groupoids $S$ satisfying the conditions 2 and 3 for pogroupoids as well as the idempotent identity $x^2=x$, then we obtain a class of groupoids which is closed under the direct product $S \times T$ defined by taking $S \times T$ to have the underlying set $S \times T$ and an operation $(s,t)\cdot(s',t')=(ss',tt')$. The class of pogroupoids is not closed under this natural direct product. It follows easily that the class of pogroupoids cannot be defined by means of identities of the type $w(x_1, \cdots, x_n)=v(x_1, \cdots, x_n)$, where the latter expressions are words in the variables $x_1, \ , x_n$. This points out the nonalgebraic nature of condition 1, and thus presumably the limitations of this approach in reducing the class of posets to a category of groupoids specified by equations.

If we consider the class of groupoids satisfying conditions 2 and 3 as well as the idempotent identity, then we may associate with each such groupoid a partially ordered set in the manner of the theorem. This allows us to construct a direct product $S \cdot T$ of pogroupoids $S$ and $T$ which is again a pogroupoid and which corresponds by the theorem to the product of posets. This product has many of the standard properties of direct products. Some properties are preserved, such as connectedness, the lattice property, etcetera. Identities of the type $w(x_1, \cdots, x_n)=v(x_1, \cdots, x_n)$ are in general not preserved unless they hold for the class of all posets. Hence the properties which are preserved and which do not hold for the class of all posets cannot be defined by the addition of identities to the identities 2, 3 and the idempotent identity. In conclusion then, the method works quite well as far as it goes. For further refinements one might begin by considering sets with two binary operations such as is done in the case of lattices for example.

PROOF. Suppose that $S$ is a pogroupoid. Then define an order relation $\leq$ on $S$ according to the rule: $y \leq x$ if and only if $xy=x$.
We claim that $(S, \leq)$ is a poset. Since $xx=x$, it follows that $x \leq x$. Suppose that $x \leq y$ and $y \leq x$. Then $xy=x$ and $yx=y$. Hence, $x=xy=x(yx)=yx=y$ by condition 2.

Finally, suppose that $x \leq y$ and $y \leq z$. Then $yx=y$ and $zy=z$. It follows that $z=zy=(zy)(yx)=(zy)x=zx$ by condition 3, i.e., $x \leq z$. Hence $(S, \leq)$ is a poset as claimed.

Now suppose that $(S, \leq)$ is a poset. Then we define a binary operation on $S$

according to the rule $xy=x$ if $y\leq x$ and $xy=y$ otherwise.

We assert that the resulting groupoid is in fact a pogroupoid. Condition 1 is obviously satisfied.

To check condition 2 we consider three cases, viz., $x\leq y$, $y\leq x$ and $(x, y)$, where we denote the fact that $x$ and $y$ are not comparable by the sumbol $(x, y)$.

Thus, if $x\leq y$, $x(yx)=xy=y$, $yx=y$. If $y\leq x$, $x(yx)=xx=x$ and $yx=x$. Finally, if $(x, y)$, then $x(yx)=xx=x$ and $yx=x$. Hence condition 2 is also satisfied.

To verify condition 3 we consider nineteen cases, listed below. We shall perform the required computations as we list each case.

(1) $x\leq y,\ z\leq x,\ z\leq y$: $(xy)(yz)=yy=y,\ (xy)z=yz=y$;
(2) $x\leq y,\ x\leq z,\ y\leq z$: $(xy)(yz)=(xy)z$;
(3) $x\leq y,\ x\leq z,\ z\leq y$: $(xy)(yz)=yy=y,\ (xy)z=yz=y$;
(4) $x\leq y,\ z\leq x,\ z\leq y$: $(xy)(yz)=yy=y,\ (xy)z=yz=y$;
(5) $y\leq x,\ x\leq z,\ y\leq z$: $(xy)(yz)=(xy)z$;
(6) $z\leq y,\ y\leq x,\ z\leq x$: $(xy)(yz)=xy=x,\ (xy)z=xz=x$;
(7) $x\leq y,\ x\leq z,\ (y, z)$: $(xy)(yz)=(xy)z$;
(8) $y\leq x,\ y\leq z,\ (x, z)$: $(xy)(yz)=(xy)z$;
(9) $z\leq x,\ z\leq y,\ (x, y)$: $(xy)(yz)=yy=y,\ (xy)z=yz=y$;
(10) $y\leq x,\ z\leq x,\ (y, z)$: $(xy)(yz)=(xy)z$;
(11) $x\leq y,\ z\leq y,\ (x, z)$: $(xy)(yz)=yy=y,\ (xy)z=yz=y$;
(12) $x\leq z,\ y\leq z,\ (x, y)$: $(xy)(yz)=(xy)z$;
(13) $x\leq y,\ (x, z),\ (y, z)$: $(xy)(yz)=(xy)z$;
(14) $y\leq x,\ (x, z),\ (y, z)$: $(xy)(yz)=(xy)z$;
(15) $y\leq z,\ (x, y),\ (x, z)$: $(xy)(yz)=(xy)z$;
(16) $z\leq y,\ (x, y),\ (x, z)$: $(xy)(yz)=yy=y,\ (xy)z=yz=y$;
(17) $z\leq x,\ (x, y),\ (y, z)$: $(xy)(yz)=(xy)z$;
(18) $x\leq z,\ (x, y),\ (y, z)$: $(xy)(yz)=(xy)z$;
(19) $(x, y),\ (x, z),\ (y, z)$: $(xy)(yz)=(xy)z$.

Hence it follows that condition 3 is indeed satisfied and the groupoid $S$ defined above is in fact a pogroupoid.

Now suppose $f$: $S\rightarrow T$ is a homomorphism of pogroupoids, i.e., $f(xy)=f(x)f(y)$. Then, if we consider the corresponding posets $(S, \leq)$ and $(T, \leq)$ it follows that $f$ is a regular order preserving function.

Indeed, if $y\leq x$, then $xy=x$, whence $f(xy)=f(x)f(y)=f(x)$, i.e., $f(y)\leq f(x)$.

Also, if the situation $(x, y)$ holds then $xy=y$ and $yx=x$, so that $f(x)f(y)=f(y)$ and $f(y)f(x)=f(x)$, so that we either have $f(x)=f(y)$ or $(f(x), f(y))$.

Conversely, if $f$ is a regular order preserving function from the poset$(S, \leq)$ to the poset $(T, \leq)$ and if $S$ and $T$ are the associated pogroupoids, then $f(xy)=f(x)f(y)$, i.e., $f$: $S \rightarrow T$ is a homomorphism.

Indeed, if $y \leq x$, then $f(y) \leq f(x)$, and thus $f(x)f(y)=f(x)=f(xy)$. If $(x, y)$, then $f(x)=f(y)$ or $(f(x), f(y))$ whence $f(x)f(y)=f(y)=f(xy)$.

The required functorial isomorphism is now the obvious one: $F((S, \leq))=S$, $G(S)=(S, \leq)$, $F(f)=f$, $G(f)=f$. The proof of theorem 1 is now complete.

**Some properties of pogroupoids**

In this section we prove a few propositions connecting properties of pogroupoids with properties of posets.

PROPOSITION 1. *A poset $(S, \leq)$ is totally ordered if and only if its associated pogroupoid S is commutative.*

PROOF. Suppose that $x, y \in S$. Then $xy=yx \in \{x, y\}$, whence $x \leq y$ or $y \leq x$, i.e., $(S, \leq)$ is totally ordered. If $(S, \leq)$ is totally ordered, then $x, y \in S$ implies $x \leq y$ or $y \leq x$ and hence in any case $xy=yx$.

PROPOSITION 2. *Suppose that S is a groupoid such that $xy=yx$, $x(yx)=yx$ and $(xy)(yz)=(xy)z$. Then also $(xy)z=x(yz)$.*

PROOF. Let $x, y, z \in S$, then $(xy)z=(xy)(yz)=(yx)(zy)=(zy)(yx)=(zy)x=x(zy)$ $=x(yz)$, and the proposition follows.

COROLLARY. *If S is a commutative pogroupoid then S is associative.*

PROPOSITION 3. *A poset $(S, \leq)$ has the property that for x, y and z, $(x, y)$, $(y, z)$ implies $x=z$ or $(x, z)$, if and only if its associated pogroupoid is associative.*

PROOF. Assume that $(S, \leq)$ is a poset such that incomparability is transitive. We take $x, y$ and $z$ elements of $S$ and we consider the expression $x(yz)$. There are three possibilities, viz., $x(yz)=x$, $x(yz)=y$, $x(yz)=z$., If $x(yz)=x$ then $yz \leq x$. If $yz=y$, then $z \leq y \leq x$, whence$(xy)z=x$. If $y \leq z$, then $y \leq z \leq x$ and $(xy)z=x$. If $(y, z)$, then $x(yz)=xz=x$, whence $z \leq x$ and so $(x, y)$ is impossible since then $(x, z)$ by transitivity of incomparability. If $x \leq y$, then $z \leq x$ contradicts $(y, z)$. Hence $y \leq x$ and $(xy)z=xz=x(yz)=x$.

If $x(yz)=y$, then $yz=y$ and $xy=y$, it follows that $(xy)z=y$, i.e., $x(yz)=(xy)z$.

If $x(yz)=z$, then $yz=z$ and $xz=z$. If $y \leq z$ and $x \leq z$, then $(xy)z=z=x(yz)$. Also, if $(x, z)$ and $(y, z)$, then $(x, y)$ whence $(xy)z=z=x(yz)$. The remaining cases are $y\leq z$ and $(x,z)$ or $(y,z)$ and $x\leq z$. We have respectively $(xy)z=xz=z$ or $(xy)z=yz=z$ in the first case and $(xy)z=xz=z$ or $(xy)z=yz=z$ in the second case.

Hence it follows that $S$ is an associative pogroupoid.

On the other hand, assume that $x(yz)=(xy)z$ for all $x,y,z\in S$. Suppose that $(x,y)$ and $(y,z)$. Hence $(xy)z=yz=z$ and $x(yz)=xz=z$. Thus, we cannot have $z\leq x$. Also, $(zy)x=yx=x$ and $z(yx)=zx=x$, so that we cannot have $x\leq z$. It follows that $(x,z)$, i.e., incomparability is a transitive relation. The proposition follows.

Typically, if the poset $(S,\leq)$ is not connected, then we can decompose $(S,\leq)$ into the disjoint union of two posets $S=A\cup B$, $A\cap B=\phi$, with $x\in A$, $y\in B$ implying $(x,y)$. Thus, suppose $y,z\in B$. If $S$ is an associative pogroupoid, then $(x,y)$, $(z,x)$ implies $(z,y)$, whence $B$ is a union of points. Similarly $A$ is a union of points. Thus $(S,\leq)$ is a union of points, i.e., $(x,y)$ if $x\neq y$. The associated pogroupoid is the right semigroup with multiplication $xy=y$.

COROLLARY. *If $S$ is an associative pogroupoid which is not the right semigroup, then $(S,\leq)$ is a connected poset.*

**Connectedness**

Suppose $S$ is any groupoid whatsoever. Then *a subset $A$ of $S$ is connected in a subset $B$ of $S$,* if given any $x,y\in A$, there is a finite subset $\{a_1,\cdots,a_k\}$ of $B$ such that:

$$xa_1=a_1x,\ a_ia_{i+1}=a_{i+1}a_i,\ i=1,\cdots,k-1,\ ya_k=a_ky.$$

A subset $A$ of $S$ is connected if it is connected in itself.

PROPOSITION 4. *If $f:S\to T$ is a homomorphism of groupoids and if $A\subset S$ is connected in $B\subset S$, then $f(A)\subset T$ is connected in $f(B)\subset T$.*

PROOF. The proof is a trivial consequence of the fact that if $xy=yx$ in $S$, then $f(x)f(y)=f(y)f(x)$ in $T$.

COROLLARY. *The homomorphic image of a connected groupoid is itself connected.*

PROOF. Take $A=B=S$ and $f(S)=T$, the conclusion follows.

PROPOSITION 5. *The poset $(S,\leq)$ is connected if and only if its associated pogroupoid is connected.*

PROOF. If $(S,\leq)$, then we define a relation $\sim$ on $S$ by saying $x\sim y$ provided there exists a finite subset $\{a_1, \cdots, a_k\}$ of $S$ such that the following statements are simultaneously false:

$$(x, a_1),\ (a_i, a_{i+1}),\ i=1, \cdots, k-1,\ (a_k, y).$$

This is an equivalence relation which partitions $S$ into subsets which we shall call the components of $S$. Clearly, if $A$ and $B$ are different components of $S$, then $x \in A$ and $y \in B$ implies $(x, y)$. Indeed, if it is false that $(x, y)$, then taking $a_1 = y$, it follows that the following statements are simultaneously false:

$(x, y)$, $(y, y)$. Hence $x \sim y$, i.e., $x$ and $y$ are elements of the same component.

Now, in terms of pogroupoids the fact that $(x, y)$ is false, is equivalent to saying $xy = yx$. The proof of the proposition is now immediate.

COROLLARY. *If $S$ is an associative pogroupoid which is not the right semigroup, then $S$ is connected.*

A direct proof of this corollary, i.e., one that does not make use of the machinery already developed and which relies only on the established identities does not seem excessively easy.

LEMMA. *If $S$ is an associative pogroupoid, then $xy \neq yx$, $xz \neq zx$ and $y \neq z$ implies $yz \neq zy$.*

PROOF. If $xy \neq yx$, then $xy = y$ and $yx = x$. Indeed, if $xy = x$, then $yx = y$ and $y = yx = x(yx) = xy = x$, whence $xy = yx$. Now, if $xy \neq yx$, $xz \neq zx$, then $(yx)z = xz = z = y(xz) = yz$ and $y = xy = (xz)y = z(xy) = zy$. The lemma follows.

PROPOSITION 6. *If $S$ is a connected associative pogroupoid, then given any elements $x$ and $y$ there is an element $z$ such that $xz = zx$ and $yz = zy$.*

PROOF. Suppose $x$ and $y$ are elements for which no element $z$ with the required properties exists. Let $\{a_1, \cdots, a_k\}$ be a minimal set of elements connecting $x$ and $y$. Then, without loss of generality we may take $k = 2$ by identifying $y$ with $a_3$ if necessary. Thus, $xa_1 = a_1x$, $a_1a_2 = a_2a_1$, $a_2y = ya_2$. It follows that $xa_2 \neq a_2x$ and $xy \neq yx$, whence by the lemma $a_2y \neq ya_2$ or $a_2 = y$. But then $a_2 = y$, and $a_1 = z$ is the required element.

Now suppose that $S$ is a connected and associative pogroupoid. We let $x \sim y$ if either $x = y$ or $(x, y)$. It follows that $\sim$ is an equivalence relation, which decomposes $S$ into a set of equivalence classes which we shall denote by $[x] = \{y \mid x \sim y\}$.

PROPOSITION 7. *If $S$ is a connected and associative pogroupoid, let $S/\sim$ denote* *the family of equivalence classes* $[x]$. *Then,* $S/\sim$ *becomes a commutative pogroupoid with the operation defined by* $[x]\,[y]=[xy]$.

PROOF. To show that the operation is well-defined we need to show that if $x\sim x'$ and $y\sim y'$, then also $xy\sim x'y'$. If $x=x', y=y'$, the situation is trivial. If $x=x'$ and $(y,y')$, suppose that $xy\nsim x'y'=xy'$. Then, either $xy\leq xy'$ or $xy'\leq xy$. Hence in the first case $(xy)(xy')=(xy')(xy)=xy'$. From the associative law and condition 2 we find that $(yx)y'=xy'$ and $y'(xy)=xy'$. But then $y'\leq xy$ or $xy\leq y'$. Since $(y,y')$, we have in either case $xy=x$, i.e., $y\leq x$, and thus since $(y,y')$, $y'\leq x$ or $(y',x)$. If $y'\leq x$, then $xy'=x'y'=x$ and $xy=x$ so that $xy=x'y'$. If $xy'\leq xy$, the same argument holds provided we interchange $y$ and $y'$. Since $xy\nsim x'y'=xy'$ we reach a contradication and $xy\sim x'y'$.

If $(x,x')$ and $y=y'$, suppose that $xy\nsim x'y'=x'y$. Then, $xy\leq x'y$ or $x'y\leq xy$. In the second case $(xy)(x'y)=(x'y)(xy)=xy$. We find that by the associative law and condition 2, $(xx')y=xy$ and $(x'x)y=xy$. Thus, since $(x,x')$ we have $x'y=xy$, whence $xy\sim x'y'$. The first case is similar. Since we assumed $xy\nsim x'y'$ we reach a contradiction and $xy\sim x'y'$.

If $(x,x')$ and $(y,y')$, then $x\leq y$ implies $x'\leq y$ or $y\leq x'$, and thus $x'\leq y$. We have $xy=y$, $x'y=yx'=y$, and $x'y'=x(yy')=(x'y)y'=yy'=y'$ so that $(xy,x'y')$.

Similarly for $x'\leq y$, $x\leq y'$, $y\leq x$, $y\leq x'$, $y'\leq x$, $y'\leq x'$, with the appropriate interchange of symbols. The case remaining is that where also $(x,y)$, $(x,y')$, $(x',y)$ and $(x',y')$. But then $xy=y$, $x'y'=y'$ and $(xy,\ x'y')$ as well. Hence $\sim$ is indeed well defined and $S/\sim$ becomes a pogroupoid immediately.

To show commutativity we must show that $xy\sim yx$, i.e., $xy=yx$ or $(xy,yx)$.

Now, if $x\leq y$, then $xy=yx=y$ while if $y\leq x$, then $xy=yx=x$, so that $xy\sim yx$.

If $(x,y)$, then $xy=y$ and $yx=x$ so that $(xy,yx)$, which means that $xy\sim yx$ as asserted. The proposition follows.

Of course, in the light of proposition 1 this means that $(S/\sim,\leq)$ is a totally ordered set. Now, if $[x]\leq[y]$, then $\alpha\in[x]$ and $\beta\in[y]$ yield $[\alpha\beta]=[xy]=[y]$, so that $\alpha\beta\sim\beta$, whence $\alpha\beta=\beta$ or $(\alpha\beta,\beta)$. Now, $(\alpha\beta,\beta)$ implies $(\alpha\beta)\beta=\alpha\beta=\beta$. Since presumably $\alpha\beta\neq\beta$, we have a contradiction, i.e., the case $(\alpha\beta,\beta)$ does not occur.

Since we also have $\beta\alpha\sim\beta$, we must have $\beta\alpha=\beta$ or $(\beta\alpha,\beta)$. Now $(\beta\alpha,\beta)$ implies $(\beta\alpha)\beta=\beta$ and $\beta(\beta\alpha)=\beta\alpha$. Suppose $\beta\alpha\neq\beta$. Then $\beta\alpha=\alpha$. Now by the equivalence $\beta\alpha\sim\alpha\beta$ we find that $\alpha\sim\beta$, and hence $x\sim y$, i.e., $[x]=[y]$. Thus, if $[x]\neq[y]$, we must have $\alpha\beta=\beta\alpha=\beta$, i.e., $\alpha\leq\beta$.

Finally, the equivalence classes $[x]$ are themselves pogroupoids which are also associative. Since $\alpha, \beta \in [x]$ implies $\alpha=\beta$ or $(\alpha, \beta)$, it follows that $[x]$ itself is the right semigroup and that $[x]$ as a poset consists of loose points. Proposition 7 thus has the following.

COROLLARY. *$S$ is an associative pogroupoid if and only if $(S, \leq)$ is an ordinal sum of posets $(S_i, \leq)$, $i \in I$, such that each poset $(S_i, \leq)$ is a collection of loose points, i.e., a set along with the diagonal relation.*

The converse is quite immediate, i.e., if $(S, \leq)$ is an ordinal sum of posets $(S_i, \leq)$, $i \in I$, such that each poset $(S_i, \leq)$ is a collection of loose points, then if $(x, y)$ and $(y, z)$, it follows that $x, y, z \in S_i$ for the same index $i \in I$, and thus if $x \neq z$, then $(x, z)$. Hence we have

COROLLARY 2. *$(S, \leq)$ is a poset such that for $x, y$ and $z$ $(x, y)$ and $(y, z)$ implies $x=z$ or $(x, z)$, if and only if $(S, \leq)$ is an ordinal sum of posets $(S_i, \leq)$, $i \in I$, such that each poset $(S_i, \leq)$ is a collection of loose points, i.e., a set along with the diagonal relation.*

**Other laws**

We have looked at the effect of associativity and commutativity. In this section we consider several other laws.

PROPOSITION 8. *Every pogroupoid $S$ is a flexible groupoid, i.e., $(xy)x=x(yx)$ for all $x, y \in S$.*

PROOF. Since $x(yx)=yx$, the flexible law becomes equivalent to the statement $(xy)x=yx$. Now, if $xy=x$, then $y \leq x$ and hence $yx=x$ as well. Hence $(xy)x=x=yx$. If $xy=y$, then $(xy)x=yx$ also.

COROLLARY. *If $S$ is a pogroupoid, then in its associated poset $(S, \leq)$ it is true that $xy=yx$ or $(xy, yx)$.*

PROOF. Since $(xy)(yx)=(xy)x=yx$ and $(yx)(xy)=(yx)y=xy$ the conclusion follows.

PROPOSITION 9. *Every pogroupoid $S$ is an alternative groupoid i.e., $(xx)y=x(xy)$ and $(yx)x=y(xx)$ for all $x, y \in S$.*

PROOF. Since $x^2=x$, we need to show that $xy=x(xy)$ and $yx=(yx)x$. If $xy=x$, then $xy=x=x^2=x(xy)$ and if $xy=y$, then $xy=y=xy=x(xy)$. Similarly, if $yx=x$, then $yx=x=x^2=(yx)x$, while if $yx=y$, then $yx=y=(yx)x$.

The class of groupoids $S$ satisfying the following conditions 1: $x^2=x$; 2: $(xy)x=x(yx)$; 3: $(xx)y=x(xy)$; 4: $y(xx)=(yx)x$; 5: $x(yx)=yx$; 6: $(xy)(yz)=(xy)z$, comes relatively close to describing the class of pogroupoids. They are not the same class however. To see this is actually quite easy. In fact, if we let $w(x_1, \cdots, x_n)$ be any word in the variables, $x_1, \cdots, x_n$, where variables may be absent, and where a word consists of a sequence of variables and parentheses so that for particular values in the groupoid $S$ the expression is always defined, then a typical identity for a groupoid has the form

$$w(x_1, \cdots, x_n)=v(x_1, \cdots, x_n).$$

Clearly, if $I$ is a set of identities satisfied by groupoids $S$ and $T$, then if we define $S\times T$ by taking the product componentwise, i.e., $(s,t)\cdot(s',t')=(ss',tt')$, then $S\times T$ satisfies the same set of identities.

Thus, if the pogroupoids were definable by a set of identities $I$, finite or infinite of whatever cardinality, then given pogroupoids $S$ and $T$, the direct product $S\times T$ should also be a pogroupoid.

Consider the pogroupoid $S$ corresponding to the two point poset $S=\{a,b\}$ with $a\leq b$. Then $S$ has the table $a^2=a$, $ab=ba=b^2=b$. In particular, in $S\times T$ we find that $(a,b)\cdot(b,a)=(b,b)$, whence condition 1 in the definition of pogroupoids is violated. Thus, the class of pogroupoids is not closed under this type of direct product. Later on we shall see that this problem can be adjusted. However we do have the following conclusion.

PROPOSITION 10. *The class of pogroupoids cannot be equationally defined.*

For examples demonstrating the independence of the conditions 1, 2 and 3 in the definition of pogroupoid we note the following.

Condition 1 is not a consequence of conditions 2 and 3, since then the class of pogroupoids would be equationally defined.

Condition 2 is not a consequence of conditions 1 and 3. Indeed, let $X$ be the left semigroup on at least two elements, $xy=x$. Then 1 and 3 are trivially satisfied, while if $x\neq y$, then $x(yx)\neq yx$.

Also condition 3 is not a consequence of conditions 1 and 2. Let $X=\{a,b,c\}$ be the groupoid with multiplication table:

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | $c$ |
| $b$ | $a$ | $b$ | $b$ |
| $c$ | $c$ | $b$ | $c$ |

then one checks easily that $a(ba)=ba$, $a(ca)=ca$, $b(ab)=ab$, $b(cb)=cb$, $c(ac)=ac$, $c(bc)=bc$, $(ab)(bc)\neq(ab)c$ and $xy\in\{x,y\}$ for all $x.y\in S$.

There is a sort of standard technique in isolating which posets satisfy a given identity. We give an example using only three variables $x, y$ and $z$. The law we will consider is $x(yz)=y(xz)$. For reference we will use the nineteen cases in the proof of theorem 1.

(1) $x(yz)=xy=y$, $y(xz)=yx=y$;
(2) $x(yz)=xz=z$, $y(xz)=yz=z$;
(3) $x(yz)=xy=y$, $y(xz)=yz=y$;
(4) $x(yz)=xy=y$, $y(xz)=yx=y$;
(5) $x(yz)=xz=z$, $y(xz)=yz=z$;
(6) $x(yz)=xy=x$, $y(xz)=yx=x$;
(7) $x(yz)=xz=z$, $y(xz)=yz=z$;
(8) $x(yz)=xz=z$, $y(xz)=yz=z$;
(9) $x(yz)=xy=y$, $y(xz)=yx=x$; (not equal)
(10) $x(yz)=xz=x$, $y(xz)=yx=x$;
(11) $x(yz)=xy=y$, $y(xz)=yz=y$;
(12) $x(yz)=xz=z$, $y(xz)=yz=z$;
(13) $x(yz)=xz=z$, $y(xz)=yz=z$;
(14) $x(yz)=xz=z$, $y(xz)=yz=z$;
(15) $x(yz)=xz=z$, $y(xz)=yz=z$;
(16) $x(yz)=xy=y$, $y(xz)=yz=y$;
(17) $x(yz)=xz=x$, $y(xz)=yx=x$;
(18) $x(yz)=xz=z$, $y(xz)=yz=z$;
(19) $x(yz)=xz=z$, $y(xz)=yz=z$.

Reading down the list we find that the only situation where equality fails to hold is the case where $z\leq x$, $z\leq y$ and $(x,y)$. This means simply that the poset cannot have a subset of the type $\{x,y,z\}$ with $z\leq x$, $z\leq y$ and $(x,y)$. Another way of saying this is that if $x$ and $y$ have a lower bound, then $x$ or $y$ is a lower bound for $\{x,y\}$.

PROPOSITION 11. *A pogroupoid $S$ satisfies the condition $x(yz)=y(xz)$ for all $x,y,z\in S$ if and only if it is true that if $x$ and $y$ are elements with a lower bound, then $x$ or $y$ is a lower bound for $\{x,y\}$.*

PROPOSITION 12. *If $S$ is a connected pogroupoid such that $x(yz)=y(xz)$, then in $(S,\leq)$ any two elements have an upper bound.*

PROOF. If $x\leq y$ or $y\leq x$, there is no problem. Suppose that we consider a smallest set $\{a_1,\cdots,a_k\}$ connecting $x$ and $y$. Suppose $k\geq 3$. Then, we have a set $\{x,a_1,a_2,a_3\}$, where $(x,a_2)$, since otherwise we would be able to eliminate $a_1$ from this smallest set. Now, $a_1$ cannot be a lower bound of $x$ and $a_2$, so that it must be an upper bound. But this implies that $a_2$ is a lower bound of $\{a_1,a_3\}$ with $(a_1,a_3)$, a manifest impossibility. Thus $k\leq 2$. If $k=2$, then if $y$ replaces $a_3$ we have another contradiction. Thus $k=1$, and $x$ and $y$ have an upper bound.

Given a subset $A$ of a pogroupoid $S$, let $U(A)=\{z\mid zx=xz=z$ for all $x\in A\}$.

Thus $U(A)$ is the set of upper bounds of $A$ in $(S, \leq)$.

PROPOSITION 13. *Suppose that $S$ is a pogroupoid such that $x(yz)=y(xz)$. Then for $A \subseteq S$, either $U(A)=\phi$ or $U(A)$ is a commutative pogroupoid. The converse is also true.*

PROOF. Suppose $z, w \in U(A)$, $x \in A$. Then $z(wx)=w(zx)=zw=wz$. Since $U(A)$ is itself a pogroupoid, the conclusion follows. Conversely, suppose that $x(yz) \neq y(xz)$ for some choice of $x, y$ and $z$. Then, we must be working in case (9), i.e., $z \leq x$, $z \leq y$ and $(x, y)$. Now, $\{x, y\} \subset U(z)$, and thus $U(z)$ is not a commutative pogroupoid.

Thus, the law $x(yz)=y(xz)$ identifies those posets $(S, \leq)$ such that the set of upper bounds $U(A)$ of a given subset $A$ is either empty or totally ordered.

**Direct products of pogroupoids**

Suppose that $S$ and $T$ are pogroupoids, then as we saw already, the direct product $S \times T$ of $S$ and $T$ is not in general a pogroupoid. We can however define a convenient direct product, which we shall denote by $S \cdot T$.

**Construction of $S \cdot T$ via posets**

If $(S, \leq)$ and $(T, \leq)$, then the product of these posets is defined by taking the set $S \times T$ and on it defining the partial order $(s, t) \leq (s', t')$ if and only if $s \leq s'$ and $t \leq t'$.

This yields a poset which we shall denote by $(S, \leq) \cdot (T, \leq)$. The poset $(S, \leq) \cdot (T, \leq)$ in its turn determines a pogroupoid which we shall denote by $S \cdot T$.

More generally, if $\{(S_\alpha, \leq)\ \alpha \in n\}$ is a family of posets, then we can define a poset $(S, \leq)=\prod_{\alpha \in n}(S_\alpha, \leq)$ in essentially the same fashion.

**Construction of $S \cdot T$ via groupoids**

The pogroupoids are among the groupoids $S$ satisfying the three conditions $1{:}\ x^2=x$; 2: $x(yx)=yx$ and 3: $(xy)(yz)=(xy)z$.

Since the defining conditions are words, it is clear that the direct product of elements in this class is again in this class. We shall refer to groupoids of this type as *weakly ordered groupoids* or *wogroupoids* for purposes of abbreviation.

Suppose now that $S$ is a wogroupoid. Then we define a relation $\leq$ on $S$ by letting $y \leq x$ if and only if $xy=x$. Since $x^2=x$, it follows that $x \leq x$. Also, if $x \leq y$ and $y \leq x$, then $x=xy=x(yx)=yx=y$. Finally, if $x \leq y$ and $y \leq z$, then

$z=(zy)(yx)=(zy)x=zx$, whence $x \leq z$. Hence $(S, \leq)$ is a poset. Define $G^*(S)=(S, \leq)$.

If $S$ and $T$ are wogroupoids, and if $f: S \to T$ is a homomorphism then $xy=x$ implies $f(x)f(y)=f(x)$, and thus the mapping $f: (S, \leq) \to (T, \leq)$ is order preserving. Now, if $(x, y)$ in $(S, \leq)$, then $xy \neq x$. This does not guarantee that $f(xy) \neq f(x)$, and so it is no longer true that $f: (S, \leq) \to (T, \leq)$ is a regular order preserving function. If we let $G^*(f)=f$, then we do get a contravariant functor from the category of wogroupoids and homomorphisms to the category of posets and order preserving maps.

PROPOSITION 14. *If $S$ and $T$ are wogroupoids, then*

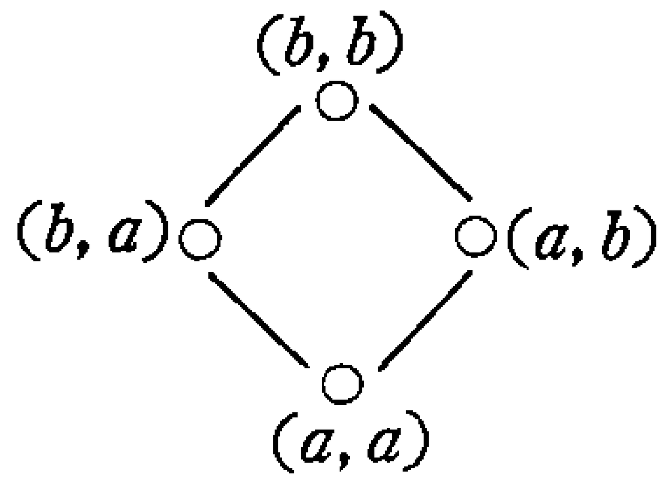
$$G^*(S \times T)=G^*(S) \cdot G^*(T).$$

PROOF. It is clear that in both cases the underlying sets are precisely the same, i.e., $S \times T$.

Now, let $(s, t) \leq (s', t')$ in $G^*(S) \cdot G^*(T)$, then $s \leq s'$ and $t \leq t'$, i.e., $s's=s'$, $t't=t'$. Hence $(s', t')(s, t)=(s's, t't)=(s', t')$ so that $(s, t) \leq (s', t')$ in $G^*(S \times T)$. On the other hand, if $(s, t) \leq (s', t')$ in $G^*(S \times T)$, then $(s', t')(s, t)=(s', t')=(s's, t't)$, so that $s's=s'$, $t't=t'$ and $(s, t) \leq (s', t')$ in $G^*(S) \cdot G^*(T)$.

Thus, in terms of the functors $G^*$ and $F$ (from theorem 1), we have $S \cdot T = FG^*(S \times T)=F(G^*(S) \cdot G^*(T))$ defined for wogroupoids, since $FG^*$ is a covariant functor from the category of wogroupoids and homomorphisms to the category of pogroupoids and homomorphisms.

Suppose that $S$ is the pogroupoid corresponding to the two point poset $S=\{a, b\}$ with $a \leq b$. Then $S$ has table $a^2=a$, $ab=ba=b^2=b$. The table for $S \cdot S$ can be constructed by observing that $G^*(S \times S)$ is the poset with Hasse diagram:




Thus, we have the table:

| $S \cdot S$ | $(a, a)$ | $(a, b)$ | $(b, a)$ | $(b, b)$ |
|---|---|---|---|---|
| $(a, a)$ | $(a, a)$ | $(a, b)$ | $(b, a)$ | $(b, b)$ |
| $(a, b)$ | $(a, b)$ | $(a, b)$ | $(b, a)^*$ | $(b, b)$ |
| $(b, a)$ | $(b, a)$ | $(a, b)^*$ | $(b, a)$ | $(b, b)$ |
| $(b, b)$ | $(b, b)$ | $(b, b)$ | $(b, b)$ | $(b, b)$ |

On the other hand, $S \times S$ has the table:

| $S \times S$ | $(a,a)$ | $(a,b)$ | $(b,a)$ | $(b,b)$ |
|---|---|---|---|---|
| $(a,a)$ | $(a,a)$ | $(a,b)$ | $(b,a)$ | $(b,b)$ |
| $(a,b)$ | $(a,b)$ | $(a,b)$ | $(b,b)^*$ | $(b,b)$ |
| $(b,a)$ | $(b,a)$ | $(b,b)^*$ | $(b,a)$ | $(b,b)$ |
| $(b,b)$ | $(b,b)$ | $(b,b)$ | $(b,b)$ | $(b,b)$ |

The effect of $FG^*$ on $S \times S$ has thus been to adjust the positions in the table marked with an asterisk.

The product $S \cdot T$ of wogroupoids has the standard properties. Thus, if $S$ and $T$ are wogroupoids and $f$: $S \to T$ is an isomorphism, then $G^*(f)$: $(S, \leq) \to (T, \leq)$ is an isomorphism of posets and thus $FG^*(f)$: $F(S, \leq) \to F(T, \leq)$ is an isomorphism of pogroupoids. As a matter of fact, on the underlying sets the mappings remain unchanged. Hence, since $S \times T$ and $T \times S$ are isomorphic by $F(s,t) = (t,s)$, $S \cdot T$ and $T \cdot S$ are isomorphic by the same map. Since $(S \times T) \times U$ and $S \times (T \times U)$ are isomorphic by $f((s,t),u) = (s,(t,u))$, $(S \cdot T) \cdot U$ and $S \cdot (T \cdot U)$ are isomorphic by the same map.

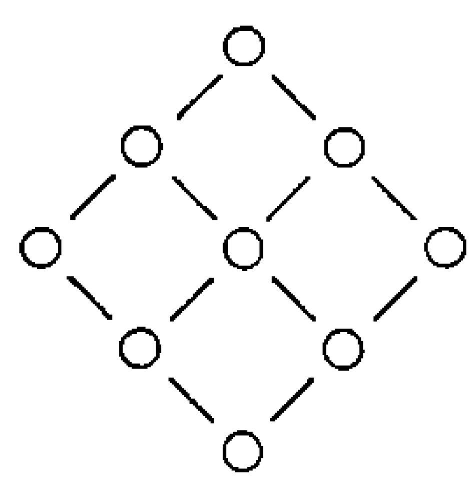
The mappings $S \times T \to S$ and $S \times T \to T$ given by projection on the components are homomorphisms, and thus induce the projection homomorphisms $S \cdot T \to S$ and $S \cdot T \to T$.

If we consider only wogroupoids with an element $e$ such that $ex = xe = x$ for all $x \in S$, i.e., we adjoin a universal lower bound, then considering only homomorphisms $f$: $S \to T$ such that $f(e) = e$, products $S \times T$ have the following universal property. Suppose that $U$ is a wogroupoid, and suppose that we have homomorphisms $f$: $S \to U$, $f'$: $U \to S$, $g$: $T \to U$, $g'$: $U \to T$ such that $g'f(S) = e$, $f'g(T) = e$, $f'f$: $S \to S$ and $g'g$: $T \to T$ are identity maps. Define $h$: $U \to S \times T$ by $h(U) = (f'(u), g'(u))$. Then $h$ is quite clearly a homomorphism. Given $(s,t) \in S \times T$, let $u = f(s)g(t)$. Then $f'(u) = se = s$, $g'(u) = et = t$, and thus $h(u) = (s,t)$, whence $h$ is a surjection. The same universal property then holds for products $S \cdot T$ of pogroupoids.

Although the product $S \cdot T$ preserves many common properties of $S$ and $T$ such as connectedness (the product of connected posets is connected), the lattice property $((x,y) \vee (u,v) = (x \vee u, y \vee v)$, $(x,y) \vee (u,v) = (x \vee u, y \vee v))$, the property of being a product of chains, etcetera, other properties are lost. Among these are commutativity, associativity and the property that the set of upper bounds of a subset is empty or a chain. The three properties we've cited are precisely those given by algebraic identities, viz., $xy=yx$, $x(yz)=(xy)z$ and $x(yz)=y(xz)$. Other identities are preserved, e.g., properties 2 and 3 themselves, $x^2=x$, the flexible property $(xy)x=x(yx)$ and the alternative properties $(xx)y=x(xy)$, $(yx)x=y(xx)$.

PROPOSITION 15. *Suppose that $w(x,y,z)=v(x,y,z)$ is an identity in three variables which holds for chains and which is preserved under products. Then this identity holds for posets in general.*

PROOF. Here we mean that the identity is to be applied to the associated pogroupoids of course. We observe that since the identity holds for chains with three elements, it also holds for posets of the type whose Hasse diagram is drawn below:



The nineteen cases are included in this diagram as an inspection will demonstrate quite readily. Since the identity must hold in each of the nineteen cases, it must hold in general.

In particular then on pogroupoids it must be a consequence of the properties defining pogroupoids. Without going into detail, it seems clear that one can prove a proposition equivalent to proposition 14 for identities involving an arbitrary finite number of variables which hold for chains and which are preserved under products.

The University of Alabama

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