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ON CERTAIN HERMITE APPROXIMATIONS

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1. Introduction

In the classical Hermite approximation, we construct the unique polynomial $p(x)$ of degree $\leq 2n+1$ such that

(1) $f(x_i)=p(x_i),\ f'(x_i)=p'(x_i),\ 0\leq i\leq n,$

where $x_0, x_1, \cdots, x_n$ are distinct points.

In this paper we are concerned with the Hermite approximation in which we construct a polynomial $p(x_1, x_2, \cdots, x_k)$ of degree $\leq 2n+2$ in $x_1, x_2, \cdots, x_k$, such that

$$(2)\quad \frac{\partial^{r_1+r_2+\cdots+r_k}}{\partial x_1^{r_1}\partial x_2^{r_2}\cdots\partial x_k^{r_k}} f(x_1, x_2, \cdots, x_k) = \frac{\partial^{r_1+r_2+\cdots+r_k}}{\partial x_1^{r_1}\partial x_2^{r_2}\cdots\partial x_k^{r_k}} p(x_1, x_2, \cdots, x_k)$$

for $0\leq r_1, r_2, \cdots, r_k\leq n$ at each vertex of the k-dimensional unit cube and

$$(3)\quad \int_0^1\int_0^1\cdots\int_0^1 f(x_1, x_2, \cdots, x_k)dx_1dx_2\cdots dx_k = \int_0^1\int_0^1\cdots\int_0^1 p(x_1, x_2, \cdots, x_k)dx_1dx_2\cdots dx_k,\quad (k\geq 1).$$

2. k-dimensional form of an imposed Hermite approximation

We construct the k-dimensional Hermite approximation formula which satisfies (2) and (3). To simplify the notation, we introduce the following. Let S_k be the set of all vertices of the k-dimensional unit cube. Let $\pi\in S_k$ and we write $\pi=(\pi(1), \pi(2), \cdots, \pi(k))$. (Note that $\pi(i)=0$ or 1.) For an independent variable x_t, we define $\pi(x_t)$ by

$$\pi(x_t)=\begin{cases} x_t & \text{if } \pi(t)=0,\\ 1-x_t & \text{if } \pi(t)=1.\end{cases}$$

We also define $|\pi(r_t)|$ by $|\pi(r_t)|=\begin{cases} r_t & \text{if } \pi(t)=1,\\ 0 & \text{if } \pi(t)=0,\end{cases}$

and define $|\pi(r_1, r_2, \cdots, r_k)|=\sum_{t=1}^{k}|\pi(r_t)|$ for $0\leq r_t\leq n$.

We write $f_{r_1, r_2, \cdots, r_k}(y_1, y_2, \cdots, y_k)$ to denote

$$\frac{\partial^{r_1+r_2+\cdots+r_k}}{\partial x_1^{r_1}\partial x_2^{r_2}\cdots\partial x_k^{r_k}} f(x_1, x_2, \cdots, x_k) \text{ at } x_i=y_i\ (i=1, 2, \cdots, k).$$

Using the above notations, we can introduce a polynomial $p_1(x_1, x_2, \cdots, x_k)$ of degree $\leq 2n+1$ in $x_1, x_2, \cdots, x_k$ by

(4) $p_1(x_1, x_2, \cdots, x_k) = \sum_{t_1, t_2, \cdots, t_k=0}^{n} [\sum_{\pi \in S_k} (-1)^{|\pi(t_1, t_2, \cdots, t_k)|} f_{t_1, t_2, \cdots, t_k}(\pi)\ q_{t_1}(\pi x_1)\, q_{t_2}(\pi x_2) \cdots$
$q_{t_k}(\pi x_k)]$, where $q_j(x)$ is a polynomial defined by (see [2, (8)])

(5) $q_j(x) = \frac{x^j}{j!}(1-x)^{n+1}(\sum_{s=0}^{n-j}\binom{n+s}{s}x^s)$.

Now we can state the following proposition.

PROPOSITION 1.

(6) $p(x_1, x_2, \cdots, x_k) = p_1(x_1, x_2, \cdots, x_k)$

$$+\frac{(\prod_{i=1}^{k}(1-x_i)x_i)^{n+1}}{(2n+3)^k\,[\binom{2n+2}{n+1}]^k}\int_0^1\int_0^1\cdots\int_0^1(f(x_1, x_2, \cdots, x_k)-p_1(x_1, x_2, \cdots, x_k))dx_1dx_2\cdots dx_k$$

is a polynomial of degree $\leq 2n+2$ *in* $x_1, x_2, \cdots, x_k$, *which satisfies* (2) *and* (3).

PROOF. We observe the integral

$\int_0^1\int_0^1\cdots\int_0^1 p(x_1, x_2, \cdots, x_k)dx_1dx_2dx_k$ of $p(x_1, x_2, \cdots, x_k)$ in (6).

Noting that $\int_0^1\int_0^1\cdots\int_0^1[\prod_{i=1}^{k}(1-x_i)x_i]^{n+1}dx_1dx_2\cdots dx_k = \frac{1}{(2n+3)^k\,[\binom{2n+2}{n+1}]^k}$, we can see that $p(x_1, x_2, \cdots, x_k)$ satisfies (3). Now we consider (2). It is clear that $\frac{\partial^{r_1+r_2+\cdots+r_R}}{\partial x_1^{r_1}\partial x_2^{r_2}\cdots\partial x_k^{r_k}}(\prod_{i=1}^{k}(1-x_i)x_i)^{n+1}$ at any π in S_k is equal to 0 for $0 \leq r_1, r_2, \cdots, r_k \leq n$. Therefore it suffices to show that

$$\frac{\partial^{r_1+r_2+\cdots+r_R}}{\partial x_1^{r_1}\partial x_2^{r_2}\cdots\partial x_k^{r_k}}p_1(x_1, x_2, \cdots, x_k) = f_{r_1, r_2, \cdots, r_k}(x_1, x_2, \cdots, x_k)$$

at each π in S_k for $0 \leq r_1, r_2 \cdots, r_k \leq n$. By observing (5), it is not difficult to see that

(7) $\frac{d^{r_i}}{dx_i^{r_i}}q_{t_i}(\pi x_i) = (-1)^{|\pi(r_i)|}\delta_{t_i}^{r_i}$, where δ_t^r is the Kronecker delta. Using (7), we can see the following:

$$\frac{\partial^{r_1+r_2+\cdots+r_k}}{\partial x_1^{r_1}\partial x_2^{r_2}\cdots\partial x_k^{r_k}}p_1(x_1, x_2, \cdots, x_k)\,|_{(x_1, x_2, \cdots, x_k)=\pi}$$

$$=\sum_{t_1,\cdots,t_k=0}^{n} \Big(\sum_{\lambda\in S_k} (-1)^{|\lambda(t_1,t_2,\cdots,t_k)|} f_{t_1,t_2,\cdots,t_k}(\lambda)\frac{d^{r_1}}{dx_1^{r_1}}q_{t_1}(\lambda x_1)\cdots\frac{d^{r_k}}{dx_k^{r_k}}q_{t_k}(\lambda x_k)$$

$$=\sum_{t_1,\cdots,t_k=0}^{n} \sum_{\lambda\in S_k} (-1)^{|\lambda(t_1,t_2,\cdots,t_k)|} f_{t_1,t_2,\cdots,t_k}(\lambda)\ (-1)^{|\lambda(r_1)|+|\lambda(r_2)|+\cdots+|\lambda(r_k)|}\prod_{i=1}^{k}\delta_{t_i}^{r_i}$$

$$=(-1)^{|\pi(r_1,r_2,\cdots,r_k)|} f_{r_1,r_2,\cdots,r_k}(\pi).(-1)^{|\pi(r_1,r_2,\cdots,r_k)|}$$

$$=f_{r_1,r_2,\cdots,r_k}(\pi).$$

This proves the proposition.

3. Error analysis

First we obtain the error of $f-p$ when $k=1$. We suppose that $f^{(2n+2)}(x)$ exists at each point of $(0,1)$. Following the argument in the proof of Theorem 3.1.1 [1, p. 56], we set $\frac{f(x)-p(x)}{x^{n+1}(x-1)^{n+1}}=K(x)$, for a fixed x such that $0\neq x\neq 1$. Consider

$$w(t)=f(t)-p(t)-(t(t-1))^{n+1}K(x)$$

$$=f(t)-p_1(t)-\frac{(t(1-t))^{n+1}}{(2n+3)\binom{2n+2}{n+1}}\int_0^1(f(y)-p_1(y))dy-(t(t-1))^{n+1}K(x).$$

$w(t)$ vanishes at $t=0$, 1 and x. It is not difficult to show that $w^{(n)}(t)$ vanishes at $n+3$ points including $t=0,1$; and hence applying the generalized Rolle's Theorem 1.6.3 [1], there is a point λ such that $w^{(2n+2)}(\lambda)=0$. By computation, we obtain $K(x)$ as

$$K(x)=\frac{1}{(2n+2)!}f^{(2n+2)}(\lambda)-\frac{(-1)^{n+1}}{(2n+3)\binom{2n+2}{n+1}}\int_0^1(f(y)-p_1(y))dy.\quad \lambda\in(0,1).$$

Thus the error of $f-p$ (which will be denoted by $E(f-p)$) is given by $E(f-p)=(x(x-1))^{n+1}K(x)$. Similarly, we can obtain that $E(f(x,y)-p(x,y))=E(f(x,y)-p_1(x,y))-(p(x,y)-p_1(x,y))$, where $E(f(x,y)-p_1(x,y))$ is equal to (13) in [2]. (We assumed that $f_{2n+2,2n+2}(x,y)$ exists at each point (x,y) in $(0,1)\times(0,1)$). To give an explicit form of the k-dimensional error term of $f-p$, we review (3.1) and (3.2) of Stancu [3, p. 138]: $R(f)=R_1(T_2(f))+R_2(T_1(f))-R_2(R_1(f))$. We rewrite this as $R(f)=R_1(f)+R_2(f)-R_2R_1(f)$ or equivalently, $R=R_1+R_2-R_2R_1$, which leads inductively to, for $K=\{1,2,\cdots,k\}$, $R=\sum_{i=1}^{k}R_i-\sum_{\substack{i>j\\ i,j\in K}}R_iR_j+\sum_{\substack{i>j>m\\ i,j,m\in K}}R_iR_jR_m-\cdots$

$+(-1)^{k+1}R_kR_{k-1}\cdots R_3R_2R_1$. We define R_i as $R_i=\frac{(x_i(x_i-1))^{n+1}}{(2n+2)!}\frac{\partial^{2n+2}}{\partial x_i^{2n+2}}$ and R_if means the value of R_if at $(x_1, x_2, \cdots, x_{i-1}, \lambda_i, x_{i+1}, x_{i+2}, \cdots, x_k)$ for $\lambda_i \in (0,1)$.

Finally, the error term of $f-p$ is given by (see (4) and (6) for p and p_1)

$E(f-p)=R(f)-(p-p_1)=R(f)-(p(x_1,x_2,\cdots,x_k)-p_1(x_1,x_2,\cdots,x_k))$, where $R_iR_jf=R_i(R_j(f))$. (In the above, we assumed that $f_{2n+2,2n+2,\cdots,2n+2}(x_1,x_2,\cdots,x_k)$ exists at each point $(x_1,x_2,\cdots,x_k)$ in $(0,1)\times(0,1)\times\cdots\times(0,1)$.)

4. The uniqueness

PROPOSITION 2. *There is no polynomial $q(x_1,x_2,\cdots,x_k)$ other than $p(x_1,x_2,\cdots,x_k)$ of* (6) *of degree $\leq 2n+2$ in $x_1,x_2,\cdots,x_k$, such that $q(x_1,x_2,\cdots,x_k)$ satisfies* (2) *and* (3).

PROOF. We prove the proposition for $k=1$. Suppose that $q(x)$ is a polynomial of degree $\leq 2n+2$ such that $q(x)$ satisfies (2) and (3). Consider $F(x)=p(x)-q(x)$, which takes the form $F(x)=c_{n+1}x^{n+1}+c_{n+2}x^{n+2}+\cdots+c_{2n+2}x^{2n+2}$. From (2) and (3), we obtain a system of linear equations:

$$(8)\ \sum_{i=1}^{n+2}\frac{c_{n+i}}{n+1+i}=0,\ \sum_{i=1}^{n+2}c_{n+i}=0,\ \sum_{i=1}^{n+2}(n+i)c_{n+i}=0,\cdots,\ \frac{(n+1)!}{1!}c_{n+1}+\frac{(n+2)!}{2!}c_{n+2}+\cdots+\frac{(2n+2)!}{(n+2)!}c_{2n+2}=0.$$

These $n+2$ equations (8) with $n+2$ unknowns $c_i(i=n+1,n+2,\cdots,2n+2)$ have a non-trivial solution iff the determinant $|C|=0$, where

$$C=\begin{bmatrix} 1/n+2 & 1/n+3 & \cdots & 1/2n+3 \\ 1 & 1 & \cdots & 1 \\ n+1 & n+2 & \cdots & 2n+2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ (n+1)! & \frac{(n+2)!}{2!} & \cdots & \frac{(2n+2)!}{(n+2)!} \end{bmatrix}.$$

But by computation, we have $|C|=\prod_{m=n+1}^{1}(m!)\frac{(n+1)!}{(2n+3)!}\neq 0$. Therefore, we conclude that $c_{n+i}=0$ $(i=1,2,\cdots,n+2)$, and hence $F(x)=0$. The foregoing argument is applicable to the proof in case of $k\geq 2$, and we omit the proof for $k\geq 2$.

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