Bull. Korean Math. Soc.
Vol. 13, No. 2, 1976.

FIXED POINT THEORY ON FIBERINGS[1]

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Given a function f from a space X into itself, any question which inquires into the existence, nature and number of fixed points is called fixed point theory.

In this article, we will focus on results which require X to be a fairly reasonable space such as a compact polyhedron or a compact connected metric ANR and f a continuous function. We first discuss fixed point theory, and then results concerning fiber preserving mappings.

1. Fixed point theory.

A. L. E. J. Brouwer's Theorem.

One of the earliest theorems on fixed point theory is the following Brouwer's theorem.

THEOREM ([2]). *For $n\geqslant 0$ any map f from $D^n=\{x\in R^n \mid \|x\|\leqslant 1\}$ into itself has a fixed point.*

If there is no fixed point, then the boundary ∂D^n $(=S^{n-1})$ is a retract of D^n that is impossible by an easy application of algebraic topology.

B. The Lefschetz Theorem.

As a generalization of Brouwer's theorem, the Lefschetz fixed point theorem is one of the most useful tools in fixed point theory.

THEOREM ([18], [13]). *Let X be a compact ANR and f: $X\to X$ be a con tinuous map such that $L(f;F)\neq 0$ for a field F. Then every map homotopic to f has a fixed point. Here $L(f:F)$ is defined as*

(1) The essence of this survey article was delivered as an invited address at the Seoul National University and the Jeonbug National University in August 1976 while the author was staying in Korea by an invitation of the Korean Mathematical Society and the Department of Science and Technology, Republic of Korea.

$$\sum_k (-1)^k \, Trace\, f_i^*, \qquad f_i^*: H^k(X;F) \longrightarrow H^k(X;F).$$

We note that if f is the identity map, then $L(id, F)$ is the Euler–Poincaré characteristic number $\chi(X)$, and that the converse statement of the theorem is false because if we choose a space X whose Euler–Poincaré number is zero, then $L(id, X)=0$ and every point is a fixed point. Thus if a space X is Q-acyclic (Q=rational field) i.e., $H^p(X;Q)=0$ for all $p\neq 0$ and $H^p(X;Q)\cong Q$, then any map on X has a fixed point. Furthermore, any map on a real projective space RP^{2n}, a complex projective space CP^{2n}, a quaternionic projective space HP^{2n}, and the Hilbert cube I^∞ has a fixed point.

C. Local index theory ([3], [4]).

Another useful tool in fixed point theory is the local form of Lefschetz theorem.

Let $f: X\rightarrow X$ be a map from a compact ANR space X into itself. If U is open in X and f has no fixed points in $\dot{U}=\mathrm{cl}(U)-\mathrm{Int}(U)$, then we say (X, f, U) is admissible. For an admissible triple (X, f, U), assign a number $i(X, f, U)\in Q$ with the following properties: (there exists a unique such index)

(1) (Localization). If (X, f, U) and (X, g, U) are admissible and $f=g$ on $\mathrm{cl}(U)$, then $i(X, f, U)=i(X, g, U)$.

(2) (Homotopy). If f_0, f_1 are homotopic by a homotopy f_t and (X, f_t, U) is admissible for each t, then $i(X, f_0, U)=i(X, f_1, U)$

(3) (Additive). If (X, f, U) is admissible and $U_1 \cdots\cdots U_s$ are mutually disjoint open sets in U and f has no fixed points in $U-(\cup_{j=1}^s U_j)$, then $i(X, f, U) =\sum_{j=1}^s i(X, f, U_j)$. In particular, if f has no fixed points in U, $i(X, f, U)=0$.

(4) (Normalization). If (X, f, X) is admissible, then $i(X, f, X)=L(f, F)$. (Use a suitable cohomology theory for $L(f, F)$.)

(5) (Commutativity). If $f: X\rightarrow Y$ g: $Y\rightarrow X$ and $(X, g\circ f, U)$ is admissible, then $i(X, g\circ f, U)=i(Y, f\circ g, g^{-1}(U))$.

THEOREM ([3]). *If $i(X, f, U)\neq 0$, then f has a fixed point in U.*

D. The Nielsen number ([19]).

The Nielsen number for a map f: $X\rightarrow X$ of a compact ANR space is a non-negative integer $N(f)$ that is a lower bound for the number of fixed points of every map that is homotopic to f. Let $\Phi(f, X)=\{x \in X \mid f(x)=x\}$. Then Φ is a compact space. We divide Φ by an equivalence relation given by $x_0 \sim x_1$, $x_0, x_1 \in \Phi$ iff there is a path c: $I \to X$ from x_0 to x_1 such that c is homotopic (with ends fixed) to $f(c)$. Then the equivalence classes are a finite number of disjoint sets $F_1, F_2, \ldots, F_n$. If f is the identity map or the space is simply connected, then there is only one class provided $\Phi \neq \phi$. We choose open set U_j such that $U_j \supset F_j$ and $\mathrm{cl}(U_j) \cap \Phi = F_j$ for each j. Then the Nielsen number is defined to be the number of Nielsen classes F_j such that $i(F_j) \equiv i(X, f, U_j) \neq 0$ (see C).

THEOREMS ([3], [4], [15], [19], [22]).

(1) *Any map f: $X \to X$ of a compact ANR space has at least $N(f)$ fixed points.*

(2) *If f and g are homotopic in X, then $N(f)=N(g)$.*

(3) *Since $L(f)=i(X, f, X)=\sum_{j=1}^{n} i(X, f, U_j)$, $U_j \supset F_j$, $N(f)=0$ implies $L(f)=0$.*

However, there are manifolds in all dimensions and maps on them such that $L(f)=0$ but $N(f) \geqslant 2$ (by B. McCord).

(4) *Suppose X is simply connected. If $L(f)=0$, then $N(f)=0$, and if $L(f) \neq 0$, then $N(f)=1$ (because there is only one Nielsen class).*

(5) *In general, if $L(f) \neq 0$, then $N(f) \geqslant 1$.*

(6) *$N(f) \leqslant$ the Reidemeister number of $f, R(f)$. The Reidemeister number of f is the number of equivalence classes of the fundamental group $\Pi_1(X)$ divided by a realation given by $\alpha \sim \beta$, $(\alpha, \beta \in \Pi_1(X))$ iff there is an element $\gamma \in \Pi_1(X)$ such that, $\alpha=\gamma\beta f_*(\gamma^{-1})$. If $\Pi_1(X)$ is abelian, then $R(f)$ is the cardinality of coker$(1-f_*)$.*

E. Jiang subgroups and Jiang spaces.

To obtain a sufficient condition for $N(f)$ to be zero when $L(f)=0$, (see D (3)), and to establish bounds on $N(f)$ when $L(f) \neq 0$, Jiang [19] and Gottlieb [12] sutdied a subgroup of $\Pi_1(X)$ that is called the Jiang subgroup $T(f)$.

Let x_0 be a base point of a compact ANR space X. An element $\alpha \in \Pi_1(X, f(x_0))$ is said to be in the Jiang subgroup $T(f, x_0)$ if there is a map H: $X \times$

$[0,1]\to X$ such that $H(x,0)=f(x)=H(x,1)$ and the loop $H(x_0,-)$ represents α.

THEOREMS ([12], [19]).

(1) *If X is a path connected space, then $T(f,x_0)\cong T(f,x_1)$, x_0, $x_1\in X$. That is, the Jiang subgroup is independent, up to isomorphism, of the choice of base point.*

Let $T(X)$ denote the Jiang subgroup for the identity map id: $X\to X$. Then we have

(2) *$T(X)=T(id)\subset T(f)\subset \Pi_1(X)$ for any map f: $X\to X$.*

(3) *If f and g are homotopic, then $T(f)\cong T(g)$.*

(4) *$T(X)\subseteq P(X,x_0)\subset Z(\Pi_1(X))$, (=the center of $\Pi_1(X)$), where $P(X,x_0)$ is the set of elements that act trivially on all homotopy groups $\Pi_n(X)$. Thus (a) if X is a simply connected polyhedron that is not the homotopy type of S^1, then $T(X)=1$ since $\Pi_1(X)$ has no center; (b) Since $\Pi_1(RP^{2n})$ does not act trivially on $\Pi_{2n}(RP^{2n})$, $T(PR^{2n})=1$ for all $n>0$;* (c) *If X is a closed 2-manifold that is neither torus T^2 nor Klein bottle K^2, then $T(X)=1$ because $\Pi_1(X)$ has trivial center if X is not one of RP^2, T^2, and K^2.*

(5) *If X is a spherical space (e.g. Eilenberg-MacLane space), then $T(X)=Z(\Pi_1(X))$. Thus $T(K^2)=Z(\Pi_1(K^2))$.*

(6) *$T(X\times Y)\cong T(X)\oplus T(Y)$. However, the Jiang subgroups do not behave well with respect to a map f: $X\to Y$, i.e., $f_*(T(X))$ may not be a subgroup of $T(Y)$*

A space X is called a Jiang space if $T(X)=\Pi_1(X)$. The fundamental group of a Jiang space is necessarily abelian.

(7) *All H-spaces, the quotient space of a topological group with respect to a connected compact Lie subgroup, generalized lens spaces (see Section 2), and odd dimensional real projective spaces RP^{2n+1} are Jiang spaces.*

(8) *If a space is a Jiang space, then the following are true (true even if $T(f,x_0)=\Pi_1(X,x_0)$).*

(*a*) *All the Nielsen fixed point classes $F_1,\cdots,F_n$ of f: $X\to X$ have the same index, i.e., $i(F_1)=\cdots=i(F_n)=k$.*

(*b*) *Therefore we have $L(f,Q)=\sum_{j=1}^{n} i(F_j)=kN(f)$.*

(*c*) *$L(f)=0$ implies $N(f)=0$ (see $D(3)$ and $D(4)$).*

(*d*) *If* $L(f)\neq 0$ *and* X *is a Jiang space, then* $N(f)=R(f)$ (*see* $D(6)$).

F. Converse of Lefschetz fixed point theorom.

A compact metric ANR space X is called a Wecken space [22] if for any map $f\colon X\to X$ there exists a map g that is homotopic to the given map f and g has precisely $N(f)$ fixed points.

THEOREMS.

(1) *Every finite polyhedron* K *with property that* $st\sigma-\sigma$ *is connected for every* 0 *or* 1*-simplex* σ *of* K *is a Wecken space* [22].

(2) *Every connected finite polyhedron, which contains a* 3*-simplex and has the property that* $\partial st(v)$ *is connected for every vertex* v, *is a Wecken space* [21].

(3) *Any topological manifold of dimension* $\geqslant 3$ (*with or without boundary*) *is a Wecken space* [5].

(4) (*Converse of Lefschetz theorem*). *Let* X *be a Wecken space. If* X *is also a Jiang space and* $L(f)=0$, *then* f *is homotopic to a map that has no fixed points. Thus, if* f *is a map of a compact topological manifold* M *of dimension* $\geqslant 3$, *then* f *is homotopic to a map that has exactly* $N(f)$ *fixed points, and if* M *is also a Jiang space and* $L(f)=0$, *then* f *is homotopic to a fixed point-free map* (*Fuller*(1954)*and Fadell*(1965)).

Now we can state the Lefschetz fixed point theorem for Wecken and Jiang spaces.

(5) (*Lefschetz theorem*). *If a compact connected ANR space is a Wecken and Jiang space. then* $L(f)\neq 0$ *iff every map homotopic to* f *has a fixed point.*

(6) *If a space* X *is a Wecken space, then* X *admits a fixed point-free map homotopic to the identity iff* $\chi(X)=0$ *since* $\chi(X)=L(id_x)$.

For example, n- dimensional torus T^n is a Wecken and Jiang space and $\chi(T^n)=0$. Therefore there is a map that is homotopic to the identity and has no fixed points. (This is, of course, trivial.)

G. The fixed point property.

A space X has the fixed point property (f. p. p) if every map $f\colon X\to X$ has a fixed point. We have mentioned some such spaces in B.

THEOREMS.

(1) *If X is a Wecken space, then X has a f.p.p. iff $N(f)\neq 0$ for every map $f:X\to X$.*

(2) *If X is a Wecken-Jiang space, then X has a f.p.p. iff $L(f)\neq 0$ for every map $f:X\to X$.*

(3) *If X is a compact ANR space and Y is a Wecken space which has a f.p.p., then the mapping cylinder $M(f)$ for a map $f:X\to Y$ has a f.p.p.*

(4) *If X is a Wecken space which has a f.p.p., then $X\times[0,1]$ has a f.p.p. since $X\times[0,1]=M(id)$.*

(5) *If A, B have a f.p.p., then the wedge product $A\vee B$ has a f.p.p.*

(6) *We know that CP^{2n} has a f.p.p. We can show that the suspension ΣCP^{2n} of CP^{2n} has a f.p.p.*

However, there are examples that show (4), (5) *and* (6) *are not true in general* (*see* [5], *p.* 148~150).

(7) *Bredon* [1] *considered a space $X_\alpha=S^k \underset{\alpha}{\cup} D^{2m}$, identified by a non-trivial element $\alpha\in \Pi_{2m-1}(S^k)$. X_α has a homotopy invariant f.p.p. provided that k is odd and $2m-k-1<k-1$. However, $X_\alpha\times X_{\alpha'}$ admits a fixed point free map if the order of α and the order of α' is relatively prime.*

(8) *Husseini* [14] *has constructed smooth manifolds M and M' that have a f.p.p., but $M\times M'$ does not have a f.p.p. It is an open problem whether or not the square $M\times M$ of a manifold M which has a f.p.p. has a f.p.p.*

2. Fixed point theory on fiberings.

Let $p\colon E\to B$ be a Hurewicz fiber map; i.e., the covering homotopy property (CHP) holds for any space or equivalently it has the path lifting property (PLP). Let E^I be the space of all continuous mapping from I to E with compact open topology, and $\Omega(E)=\{(e,\omega)\in E\times B^I \mid p(e)=\omega(0)\}$. Define $q\colon E^I\to\Omega(E)$ by $q(\omega')=(\omega'(0),\ p\omega')$. A lifting function is a continuous map $\lambda\colon \Omega(E)\to E^I$ such that $q\cdot\lambda=\mathrm{id}$ on $\Omega(E)$. It is regular if it lifts a constant path to a constant path. For example, if B is a metric space, then any fibering $p\colon E\to B$ has a regular lifting function. If $p\colon E\to B$ is a locally trivial fiber space and if B is a paracompact and T_2-space, then p has a lifting function. A fibering $p\colon E\to B$ is orientable if $\omega\in B^I$ is any loop at $b_0\in B$, then $\tilde\omega\colon p^{-1}(b_0)\to p^{-1}(b_0)$ given by $\tilde\omega(e)=\lambda(e,\omega)(1)$ for each $e\in p^{-1}(b_0)$ induces an isomorphism $\tilde{\omega}_*: H_*(p^{-1}(b_0);Z)\to H_*(p^{-1}(b_0);Z)$. Let f: $E\to E$ be a fiber preserving map; i.e., $f|_{p^{-1}(b)}$: $p^{-1}(b)\to p^{-1}(b')$, $b'\in B$. If ω: $[0,1]\to B$ is a path from b' to b, then we may define a map f_b: $p^{-1}(b)\to p^{-1}(b)$ by, for each $e\in p^{-1}(b)$, $f_b(e)=\lambda(f(e),\omega)(1)$. A fiber preserving map f also defines a map f_B: $B\to B$ such that $p\cdot f=f_B\cdot p$. We note that f_b: $p^{-1}(b)\to p^{-1}(b)$ depends, in general, on the choice of path ω. But the Lefschetz number $L(f_b)$ is independent of path ω [6].

THEOREM 2.1 ([16]). *Let p: $E\to B$ be an orientable Hurewicz fiber mapping from a compact metric n-manifold ($n\geqslant 3$) onto a compact, connected metric ANR B and f: $E\to E$ be a fiber preserving map. Suppose there is a connected ANR fiber F_b and the total space E is a Jiang space (i.e., $T(E)=\Pi_1(E)$). If either f_b: $F_b\to F_b$ or f_B: $B\to B$ is homotopic to a fixed point-free map, then f is homotopic to a fixed point-free map.*

Sketch of Proof. In the case when f_B: $B\to B$ is homotopic to a fixed point-free map, it may follow by the CHP. If g_b is homotopic to f_b and it has no fixed points, then $N(f_b)=N(g_b)=0$, hence $L(f_b)=0$. Since $L(f)=L(f_B)\cdot L(f_b)$ it follows that $L(f)=0$. Since E is a Jiang space, $L(f)=0$ implies $N(f)=0$. Therefore f is homotopic to a fixed pointfree map because E is a Wecken space. The formula $L(f)=L(f_B)\cdot L(f_b)$ can be found in [7] for a polyhedron and in [16] for a compact ANR space.

Let $S^{2n+1}=\{z=(z_0,z_1,\cdots,z_n)\in C^{n+1}|\ |z|=1\}$. If α: $S^{2n+1}\to S^{2n+1}$ is a homeomorphism given by $\alpha(z)=(z_0e^{2\pi i/p}, z_1e^{2\pi iq_1/p},\cdots,z_ne^{2\pi iq_n/p})$, $p\geq 2$ odd, where $q_1,q_2,\cdots,q_n$ are integers such that $(p,q_i)=1$, then α induces a free Z_p-action on S^{2n+1}. The orbit space S^{2n+1}/Z_p is called a generalized lens space, $L_{2n+1}(p,q_1,\cdots,q_n)$. Then we know $\Pi_1(L_{2n+1}(p,q_1,\cdots,q_n))=Z_p$. Suppose (T^k,M) is a free k-dimensional toral group action on a manifold such that the orbit space M/T^k is $L_{2n+1}(p,q_1,\cdots,q_n)$. Such actions are classified by the homotopy classes $[L_{2n+1}(p),\ CP(\infty)^k]=H^2(L_{2n+1}(p),\ Z^k)=(Z_p)^k$.

THEOREM 2.2 ([16]). *If f: $M\to M$ is an equivariant map (or bundle map) and if either f_b:$T^k(x)\to T^k(x)$ or $f|_{L_{2n+1}(p)}$:$L_{2n+1}(p)\to L_{2n+1}(p)$ is homotopic to a fixed point-free map, then f is homotopic to a fixed point-free map. Where $T^k(x)$ is the orbit through the point x. To prove this, we show that M is homeomorphic to $L_{2n+1}(d)\times T^k$ for some d that divides p and use Theorem* 2.1. *In $D(2)$ of Section* 1, *we said that if $f\sim g$: $X\to X$ of a compact ANR* *space, then* $N(f)=N(g)$. Jiang [15] improved this theorem.

THEOREM 2.3 ([15]). *Suppose* h: $X\to Y$ *(compact ANR space) is a homotopy equivalence and the diagram commutes homotopically*:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow h & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

If $\Pi_1(X)$ *is finite, then* $N(f)=N(g)$.

In 1976, Fadell improved this by using the mapping cone and a local index.

THEOREM 2.4 ([10]). *Theorem* 2.3 *is true without assuming the finiteness of the fundamental group* $\Pi_1(X)$.

We have noticed that $L(f)=L(f_B)\cdot L(f_b)$ for an orientable Hurewicz fiber map p: $E\to B$ and a fiber preserving map f: $E\to E$. We try to see whether or not a similar product formula holds for the Nielsen numbers; i.e.,

$$N(f)=N(f_B)\cdot N(f_b).$$

This formula is not true in generel. An example can be formed in the Hopf fibering $S^1\to S^3\to S^2$ [8]. To this regard, Brown and Fadell were able to prove the following theorem.

THEOREM 2.5 ([6], [8]). *Let* p: $E\to B$ *be a locally trivial fiber space with fiber* F. *Suppose* E, B, F *are connected finite polyhedra,* f: $E\to E$ *a fiber preserving map. If* (*a*) $\Pi_1(B)=\Pi_2(B)=0$ *or*

(*b*) $\Pi_1(F)=0$ *or*

(*c*) $p:E\to B$ *is trivial with* $\Pi_1(B)=0$ *or* $f=f_B\times f_b$,

then $N(f)=N(f_B)\cdot N(f_b)$ *for all* f_b, $b\in B$.

Pak [20] formulated an obstruction to the product formula for the Nielsen numbers. The obstruction number is defined algebraically and some calculations of numbers in various cases can be found in [17].

THEOREM 2.6 ([20]). *Let* p: $E\to B$ *be an orientable locally trivial fiber map. Suppose* E, B, $p^{-1}(b)$ *are compact connected ANR Jiang spaces and a fiber preserving map* f *has non-zero Lefschetz number. Then there exists a number* $P(f)$ such that $N(f)\cdot P(f)=N(f_B)\cdot N(f_b)$.

EXAMPLE ([17]). Let $S^1\to E\xrightarrow{p}CP^n$ be a circle bundle over a complex projective space CP^n. Then E corresponds to some $i \in Z \cong H^2(CP^n;Z) \cong [CP^n;CP(\infty)]$. The space E will be a Jiang space. Suppose $f\colon E \to E$ is a fiber preserving map such that $L(f) \neq 0$. If degree $(f_b)=d$, $i \in Z$, then $P(f)=\dfrac{|1-d|}{(1-k,\ |i|),}$ $k \equiv \deg(f) \bmod |i|$. Therefore, the product formula holds iff $P(f)=1$; i.e., $|1-d|=(1-k,\ |i|)$. This implies that the product formula holds for S^1-bundle over CP^n iff $|1-\deg(f_b)|$ divides $|i|$.

A fibering $p\colon E \to B$ is called injective if the inclusion $i\colon p^{-1}(b) \to E$ induces a monomorphism $i_*\colon \Pi_1(p^{-1}(b)) \to \Pi_1(E)$ for all $b \in B$. We note that the Hopf fibering $S^1 \to S^3 \to S^2$ is not injective.

THEOREM 2.7 ([17]). *Let $p\colon E \to B$ be an orientable locally trivial injective fiber map. Suppose E, B, $p^{-1}(b)$ are Jiang spaces and their fundamental groups are finitely generated abelian groups. If the exact sequence*

$0 \to \Pi_1(p^{-1}(b)) \xrightarrow{i_*} \Pi_1(E) \xrightarrow{p_*} \Pi_1(B) \to 0$ *splits and $\Pi_1(B)$ is finite and $L(f) \neq 0$, then $N(f)=N(f_B)\cdot N(f_b)$.*

COROLLARY 2.8 ([17]). *Let $p\colon E \to B$ be an orientable locally trivial injectivefiber map. Suppose E, B, $p^{-1}(b)$ are Jiang spaces and their fundamental groups are finitely generated abelian groups. If $\Pi_1(B)$ is finite and either there exists a cross section or $p^{-1}(b)$ is a retract of E, then $N(f)=N(f_B)\cdot N(f_b)$.*

EXAMPLES ([17]).

(a) Let $T^k \to E \to T^n$ be a principal T^k-bundle over T^n. For any fiber preserving map $f\colon E \to E$ we have $N(f)=N(f_{T^n})\cdot N(f_b)$.

(b) Let $S^1 \to E \to L_3(p,q)$ be a principal S^1-bundle over a 3-dimensional lens space. Here E is determined by $[f_j] \in [L_3(p,q),\ CP(\infty)] \cong H^2(L_3(p,q);Z) \cong Z_p$. Let $f\colon E \to E$ be a fiber preserving map such that $f_{b*}(1)=c_2$, $f_{B*}(\bar{l}_p)=\bar{c}_1$, where 1 generates $\Pi_1(S^1) \cong Z$ and $\bar{l}_p$ generates $\Pi_1(L_3(p,q)) \cong Z_p$. Then $N(f)=N(f_b)\cdot(1-c_1,d)$, where $d=(j,p)$. Therefore, $N(f_b)\cdot(1-c_1,d)\ P(f)=N(f_b)\cdot N(f_B)$, hence $N(f_B)=(1-c_1,d)\ P(f)$. For example, if $p=15$ and $j=10$ so that $d=5$ and $1-c_1=1-c_2=3$, then $N(f)=3\cdot(3,5)=3$ so that $N(f)\neq N(f_B)\cdot N(f_b)$. However, in this case, $P(f)=3$ and $3=N(f_B)=(1-c_1,d)P(f)=(3,5)\cdot 3$.

Note that in proving the product theorem, we had to show the independency of $N(f_b)$. Fadell and Brown [6], [8] showed it for a locally trivial fiber map and polyhedra, and Fadell [11] shows the following: If $p:E\to B$ is an orientable Hurewicz fiber map, then $N(f_b)$ is well defined; i.e., it is independent of the choice of path ω: $[0,1]\to B$ from b to b' and points $b, b' \in B$.

THEOREM 2.9 ([11]). *Let $p:E\to B$ be an orientable Hurewicz fiber map (all spaces are compact metric ANR) and f: $E\to E$ a fiber preserving map. Then there exists a locally trivial fiber map $p':E'\to B'$ and fiber preserving map f': $E'\to E'$ with E', B', $p'^{-1}(b)$ finite polyhedra such that $N(f)=N(f')$, $N(f_B)=N(f'_B)$, and $N(f_b)=N(f'_b)$. Consequently, Fadell lists the product theorems.*

THEOREM 2.10 ([11]). *Let $p:E\to B$ be an orientable Hurewicz fiber map with E and B ANR's (compact metric) and $f:E\to E$ be a fiber preserving map. Then $N(f)=N(f_B)\cdot N(f_b)$ in each of the following cases:*

(*a*) $\Pi_1(B)=\Pi_2(B)=0$ ([6], [8]).

(b) $\Pi_1(F)=0$ ([6], [8]).

(*c*) *p: $E\to B$ is fiber homotopically trivial and $\Pi_1(B)=0$* ([6], [8]).

(*d*) *There is a homotopy commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\phi} & p^{-1}(b) \\ \downarrow f & & \downarrow g \\ E & \xrightarrow{\phi} & p^{-1}(b) \end{array}$$

such that $\phi|p^{-1}(b')$ is a homotopy equivalence for each $b'\in B$ ([6], [8], [11]).

(*e*) *$\Pi_1(B)=0$ and p: $E\to B$ is injective* ([20]).

(*f*) *The sequence $0\to\Pi_1(p^{-1}(b))\to\Pi_1(E)\to\Pi_1(B)\to 0$ is split exact sequence with splitting map σ, $H=\operatorname{im}\sigma$ is normal, $\Pi_1(B)$ is all torsion and $\Pi_1(E)$ is torsion free* ([17] *for Jiang spaces*).

(*g*) *$p:E\to B$ admits a section σ: $B\to E$ such that $f\sigma=\sigma\bar{f}$ and $\Pi_1(E)$ is abelian* ([17]).

THEOREM 2.11 ([11]). *Let p: $E\to B$ be an orientable Hurewicz fiber map with E, B, $p^{-1}(b)$ connected compact metric ANR and let f: $E\to E$ be a fiber preserving map. Suppose the sequence $0\to\Pi_1(p^{-1}(b))\xrightarrow{i_*}\Pi_1(E)\xrightarrow{p_*}\Pi_1(B)\to 0$*

is exact and p_* admits a right inverse (section) σ such that if $H = inv\sigma$, then $f_*(H) \subset H$, Then $N(f) = N(f_B) \cdot N(f_b)$.

THEOREM 2.12 ([11], [17]). *Let $p: E \to B$ be an orientable injective Hurewicz fiber map with E and B connected compact metric ANR and let $f: E \to B$ be a fiber preserving map. If $f_{B*}: \Pi_1(B) \to \Pi_1(B)$ fixes only the identity, then $N(f) = N(f_B) \cdot N(f_b)$.*

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