Bull. Korean Math. Soc.
Vol. 13, No. 2, 1976.

RECENT TOPICS IN DIFFERENTIAL GEOMETRY

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It is a big honor and a great pleasure for me to be able to give a lecture at this meeting celebrating the 30th anniversary of the Korean Mathematical Society held at this eminent university.

§ 1. Riemannian geometry.

Since the discovery of analytic geometry by Descartes and Fermat and that of differential and integral calculus by Newton and Leibniz, mathematicians started the study of curves and surfaces using the techniques of analytical geometry and differential and integral calculus, that is, people started the study of the so-called *differential geometry*.

In his study of surface, Gauss (1777—1855) defined the curvature of a surface at a point P on it as follows. At a point P of a surface S, he draws the normal PN to S and considers a plane containing this normal PN. The plane cuts the surface S by a plane curve and so we can consider the curvature k of this plane curve at P. When we rotate this plane arround the normal PN, the curvature k varies and gives the maximal value k_1 and the minimal value k_2. Gauss calls the ***total curvature*** the product of k_1 and k_2 and denote it by $K=k_1k_2$.

Gauss then proves that this total curvature, which is now called *Gaussian curvature*, is invariant by any isometrical deformation of the surface. The study of properties of a surface which are invariant by isometrical deformations of the surface is now called *differential geometry on the surface*.

Riemann (1826—1866), a pupil of Gauss, generalized this idea of Gauss and started the differential goemetry in an n-dimensional metric space which is a generalization of a surface in an 3-dimensional Euclidean space. Such a space is now called a *Riemannian space* or a *Riemannian manifold* and the geometry in it a *Riemannian geometry*.

The Riemannian geometry had been studied by Christoffel (1829—1900), Ricci (1853—1925), Levi-Civita (1873—1941) and others. The method developed by Ricci and Levi-Civita to study Riemannian geometry had been called by them the ***absolute differential calculus*** but now it is called the ***Ricci Calculus*** or ***Tensor Calculus*** and it is still now a very powerful tool to study the so-called differentiable manifolds.

It is very well known that Einstein (1879—1955) established his famous *general theory of relativity* in 1916 applying this method of tensor calculus to his space-time with gravitational field.

The curvature at P with respect to a plane element passing through P of a Riemannian manifold is defined as follows. Consider all geodesics passing through P and being tangent to the plane element. Then these geodesics form a 2-dimensional surface. The Gaussian curvature of this surface is called the ***sectional curvature*** at P with respect to the plane element.

Consider a vector X at P. Then there exist n-1 mutually orthogonal vectors Y_i orthogonal to X. The sum of sectional curvatures with respect to plane elements determined by X and Y_i is independent of the choice of Y_i and is called the Ricci curvature at P with respect to X. If the ***Ricci curvature*** is independent of X the Riemannian manifold is called an ***Einstein space***.

§2. Levi-Civita's parallelism and geometry of connections.

Levi-Civita introduced in 1917 the notion of parallelism of vectors in Riemannian geometry. Assume that a Riemannian manifold is a submanifold of a higher dimensional Euclidean space. Then a vector field Y in the Riemannian manifold is a vector field which is tangent to this submanifold. Let $\overrightarrow{\mathrm{PR}}$ be the value of Y at a point Q of the manifold infinitely near to the point P. The vectors $\overrightarrow{\mathrm{PR}}$ and $\overrightarrow{\mathrm{QS}}$ are in general not on a same plane. So we project orthogonally $\overrightarrow{\mathrm{QS}}$ on the plane tangent to the submanifold at P and denote by $\overrightarrow{\mathrm{Q'S'}}$ the orthogonal projection of $\overrightarrow{\mathrm{QS}}$. If $\overrightarrow{\mathrm{PR}}$ and $\overrightarrow{\mathrm{Q'S'}}$ are parallel in the ordinary sense, then Levi-Civita says that the vector field Y is parallel along $\overrightarrow{\mathrm{PQ}}$ in the Riemannian manifold.

In Riemannian geometry a differential operator, called ***covariant differential operator*** along the vector X and denoted by ∇_X plays an important rôle. When a vector field Y is parallel along X in the above sense, this fact is represented by the equation $\nabla_X Y=0$.

Elie Cartan (1869—1951) generalized the parallelism of Levi-Civita in the following way. Consider the tangent space T_P at P and the tangent space T_Q at Q of the submanifold, then the orthogonal projection of T_Q on T_P gives a Euclidean correspondence between T_P and T_Q. Thus for a submanifold regarded as a Riemannian manifold, there exists a Euclidean correspondence between two tangent spaces at infinitely near points P and Q. Cartan calls such a manifold a *manifold with Euclidean connection.*

On the other hand, it is well known that Klein (1849—1925) defined a geometry as follows. Suppose that there are given a space S and a transformation group G in it. Then the study of properties of a subset of S which are left invariant by any transformation belonging to G is called the *geometry belonging to the group G.* According as the group is Euclidean, affine, projective or conformal the corresponding geometry is called Euclidean geometry, affine geometry, projective geometry or conformal geometry respectively.

According as the correspondence between two tangent spaces of a manifold at infinitely nearby points is Euclidean, affine, projective or conformal, Elie Cartan called the manifold, *a manifold with Euclidean connection, that with affine connection, that with projective connection* or *that with conformal connection.*

Geometries of these manifolds were studied in 1920—1940 by Weyl (1885—1955), Eisenhart (1876—1965), Veblen (1880—1960), T. Y. Thomas (1899—), J. A. Schouten (1883—1971) and others in great detail.

§ 3. Local and global properties.

Tangent, normal, tangent plane and curvature of a curve or a surface can be defined when a small part of a curve or surface is given. Properties which can be stated using these terms are said to *local.*

On the other hand, properties which can be stated only when we know the totality of a curve or a surface are said to be *global.* For example, the curvature of a plane curve is constant is a local property and that such a plane curve is a circle is a global property. It is one of very interesting problems in differential geometry to study the relations between locol properties and global properties of a curve or a surface.

Gauss and Bonnet (1819—1892) obtained a theorem stating a relation between the Gaussian curvature K and Euler characteristic $\chi(S)$ of a closed surface S, that is,

$$\frac{1}{2\pi}\int_S K dS=\chi(S),$$

dS being the surface element of S.

This theorem of Gauss and Bonnet had been generalized to a Riemannian manifold by Allendoerfer (1914—1974), Chern(陳省身, 1911—), W. Fenchel (1905—), H. Hopf (1895—1971) and Weil (1906—).

§ 4. Curvature and Betti numbers.

Now a very powerful tool to study relations between local properties and global properties of a Riemannian manifold was found by Hodge (1903—).

When a p-form ω satisfies

$$d\omega=0, \quad \delta\omega=0,$$

then ω is said to be harmonic where $d\omega$ is the differential and $\delta\omega$ the codifferential of ω. Hodge proved:

THEOREM. *In a compact and orientable Riemannian manifold the number of linearly independent (with constant coefficients) harmonic p-forms is equal to the p-th Betti number of the manifold.*

I refer you to: W. V. D. Hodge, *The theory and applications of harmonic integrals*, Cambridge University Press, 1941; G. W. de Rham (1903—), *Variétés différentiables*, Hermann, Paris, 1955; de Rham and K. Kodaira, *The harmonic integrals*, Lectures delivered at the Institute for Advanced Study, 1950.

Bochner (1899—) started in 1946 the study of relations between curvature and Betti numbers of a Riemannian manifold using the above mentionned result of Hodge.

I refer you to Yano and Bochner, *Curvature and Betti numbers*, Annals of Math. Studies, No. 32, Princeton Univ. Press, 1953; S. I. Goldberg, *Curvature and homology*, Academic Press, New York, 1962; Yano, *Intergral formulas in Riemannian geometry*, Marcel Dekker, Inc. New York, 1970.

§ 5. Complex and almost complex spaces.

When an even-dimensional manifold can be covered by a system of complex coordinate neighborhoods $\{U; Z^a\}$ in such a way that in the non-empty intersection of two coordinate neighborhoods the coordinate transformation is given by complex analytic functions, the manifold is called a *complex manifold.*

In a complex manifold, there exists a numerical tensor field

$$F=\begin{bmatrix} 0 & -E \\ E & 0 \end{bmatrix}$$

E being a $n\times n$ matrix and F satisfies

$$F^2=-I,$$

I being the unit tensor.

If there exists moreover a Riemannian metric g such that

$$g(FX, FY)=g(X, Y)$$

for any vector fields X and Y of the manifold, then the manifold is said to be *Hermitian.* (Hermite, 1822—1901).

In a Hermitian manifold

$$\Omega(X, Y)=G(FX, Y)$$

is a 2-form and a Hermitian manifold for which the differential $d\Omega$ of Ω vanishes is called a *Kaehlerian manifold* (Kaehler, 1906—). It is known that in order for a Hermitian manifold to be Kaehlerian, it is necessary and sufficient that

$$\nabla_X F=0$$

for any vector field X.

Ehresmann (1905—) considered in 1950 an even-dimensional differentiable manifold in which there is given a tensor field F of type $(1, 1)$ satisfying $F^2=-I$, and called it an *almost complex manifold.* When there is given, in an almost complex manifold, a Riemannian metric g such that $g(FX, FY) =g(X, Y)$, he calls it an *almost Hermitian manifold.* A complex manifold is of course an almost complex manifold but an almost complex manifold is not necessarily complex.

On the other hand Nijenhuis (1926—) found in 1951 that when a tensor field F of type $(1, 1)$ is given in a manifold, then the expression

$$N(X, Y)=[FX, FY]-F[FX, Y]-F[X, FY]+F^2[X, Y]$$

defines a tensor field of type $(1, 2)$, where $[X, Y]$ is defined by

$$[X, Y]f=XYf-YXf$$

for an arbitrary function f. This tensor field is now called the *Nijenhuis tensor* formed with F.

In 1957, Newlander and Nirenberg proved that in order for an almost complex space with structure tensor F to be a complex manifold with F it is necessary and sufficient that the Nijenhuis tensor formed with F vanishes identically.

An almost Hermitian manifold satisfying

$$d\Omega=0$$

is called an *almost Kaehlerian manifold* and an almost Hermitian manifold satisfying

$$(\nabla_X\Omega)(Y, Z)+(\nabla_Y\Omega)(X, Z)=0$$

is called an *almost Tachibana space* or a *nearly Kaehlerian manifold.*

When the Nijenhuis tensor of F vanishes identically almost Kaehlerian manifold and almost Tachibana space reduce both to a Kaehlerian manifold.

§6. Weyl and Bochner curvature tensors.

Weyl (1885—1955) studied conformal change $\bar{g}=\rho^2. g$ of Riemannian metrics where ρ is a positive function and found that a curvature tensor C, which is now called the *Weyl conformal curvature tensor* is invariant under a conformal change of Riemannian metrics. If a Riemannian metric is conformal to a locally Euclidean metric the Riemannian manifold is said to be *conformally flat.* Weyl proved that if a Riemannian manifold is conformally flat then the Weyl conformal curvature tensor vanishes identically and Schouten proved that the converse is also true if the dimension is greater than 3. Using the techniques of "*curvature and Betti numbers*", Bochner proved:

THEOREM. *In a conformally flat compact orientable Riemannian manifold of* *dimension* $n>3$, *if the Ricci curvature is positive definite, then we have* $B_p=0$ $(p=1, 2, \cdots, n-1)$, B_p *denoting the p-th Betti number of the manifold.*

To obtain a theorem analogous to this in a compact Kaehlerian manifold Bochner introduced a curvature tensor, which is now called the *Bochner curvature tensor* and is denoted by B and obtained a theorem similar to the above the conclusion being replaced by

$$B_{2l}=1, \quad B_{2l+1}=0.$$

§ 7. Almost contact manifold.

Consider a $(2n+1)$-dimensional manifold M^{2n+1} immersed in a $(2n+2)$-dimensional almost complex manifold. Then we can see that the manifold M^{2n+1} admits a tensor field φ of type $(1, 1)$, a vector field U and a 1-form u such that

$$\varphi^2 X=-X+u(X)U, \quad \varphi U=0, \quad u(\varphi X)=0, \quad u(U)=1$$

for an arbitrary vector field X. The set (φ, U, u) satisfying these equations is called an *almost contact structure* and a manifold with an almost contact structure an *almost contact manifold.* When there is given, in an almost contact manifold, a Riemannian metric g such that

$$g(\varphi X, \varphi Y)=g(X, Y)-u(X)u(Y), \quad u(X)=g(U, X),$$

then the manifold is called an *almost contact metric manifold.*

When an almost contact structure satisfies

$$N(X, Y)+(du)(X, Y)U=0,$$

$N(X, Y)$ being the Nijenhuis tensor formed with φ and du the differential of u, then the structure is said to be *normal.* If an almost contact metric structure satisfies

$$(du)(X, Y)=g(\varphi X, Y)$$

for arbitrary vector fields X and Y, then the structure is said to be *contact.* A normal contact metric structure is called a *Sasakian structure* and a manifold with Sasakian structure is called a *Sasakian manifold.* An odd-dimensional sphere is an example of Sasakian manifold. The Bochner curvature tensor defined in a Kaehlerian manifold can also be defined in a Sasakian manifold and is called the ***contact Bochner curvature tensor.***

Differential geometers started just a few years ago, the study of Bochner and contact Bochner curvature tensors.

§ 8. Submanifolds.

Differential geometers are now studing various kinds of submanifolds such as minimal submanifolds, submanifolds with constant mean curvature, submanifolds with parallel mean curvature vector and so on. We refer to B. Y. Chen, *geometry of submanifolds,* Marcel Dekker Inc., New York, 1973.

Here I would like to mention two topics on submanifolds.

Suppose that there is given a submanifold M^n immersed in a Kaehlerian manifold M^{2m} with almost complex structure tensor F. If the transform $FT_x(M)$ of the tangent space $T_x(M)$ of M at an arbitrary point x by F is always contained in the normal space $T_x(M)^\perp$ at x, then the submanifold is said to be *totally real* or *anti-invariant.*

To quote a typical example of results on anti-invariant submanifold, we mention

THEOREM. *Let M^n $(n\geqq 3)$ be a totally real, totally umbilical submanifold of a Kaehlerian manifold M^{2m} with vanishing Bochner curvature tensor. Then M^n is conformally flat.*

Differential geometers also defined anti-invariant submanifolds of a Sasakian manifold and are now studying them.

To study infinitesimal variations of the manifold itself, we can use the Lie derivations, but to study infinitesimal variations of the submanifold, we can not use the technique of Lie derivations. Thus the differential geometers are now trying to develop a useful technique to study infinitesimal variations of submanifolds.

I would like to mention that professor Eum and professor Ki who are present at this meeting are doing very important contributions to this problems.

Thank you very much for your attention.

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