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DUAL GROUPS OF LOCALLY COMPACT ABELIAN GROUPS

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The theory of topological groups has been studied by many mathematicians, for example, Ellis [1], Husain [2], Pontrjagin [3], etc. In particular, dual groups of topological groups are studied by Wu [4] and Bourbaki.

In this paper, we shall study about Hausdorff locally compact abelian topological groups. In order to do this, in §1 we shall prove a proposition about S-topology (Proposition 1.1), in §2 we verify that the dual group of a Hausdorff locally compact abelian topological group is Hausdorff, locally compact and abelian (Theorem 2.6) using our lemmas 2.3 and 2.4 and in §3. We prove that (a) the dual group of a compact Hausdorff abelian topological group is a discrete abelian topological group, (b) the dual group of a discrete abelian topological group is compact (Theorem 3.1) and illustrate some examples for Theorem 3.1 (Examples 3.2~3.4).

Throughout this paper we shall assume that G is a topological group without any statements.

§1. S-topologies

Let G be a topological group with topology $\mathcal{U}$. Then $G^G=\prod_{g\in G} G_g$, where G_g is equal to G, is a topological group with product topology. The product topology has the subbase which is the set of all

$$T(\{g\},\ U)=\{\, f\in G^G \,|\, f(g)\in U\,\}$$

where $g\in G$ and U an open subset of G, i.e., $U\in\mathcal{U}$.

Let S is the set of all closed subsets of G. That is, $F\in S$ if and only if the complement of $F(=F^c$ is in $\mathcal{U}$. Consider the family $\{T(M,U)=\{f\in C(G,G)\,|\, f(M)\subset U\}\}$, where $C(G,G)=\{f\in G^G \,|\, f$ is continuous$\}$ and M runs over S and U over $\mathcal{U}$. The topology which has $\{T(M,U)\}$ as a subbase is called the ***S*-*topology*** of G.

For each $g\epsilon G$, there are two continuous automorphisms r_g and l_g of G which are defined as follows:

$$r_g: \begin{array}{c} G \to G \\ \Psi \quad \Psi \\ x \to xg \end{array} \quad , \qquad l_g: \begin{array}{c} G \to G \\ \Psi \quad \Psi \\ x \to gx \end{array}$$

Then the map $\eta_r: G \to G^G$ $(g \to r_g)$ $(\eta_l: G \to G^G$ $(g \to l_g))$ will be called the *right* (*left*) *canonical embedding* into G^G, where G^G is a topological group with product topology as before. In this case, G is homeomorphic with $\eta_r(G)$ $(\eta_r(G)) \subset G^G$. Furthermore, G is homeomorphic to a subset of $C(G, G)$, is a topological group with relative topology induced from the product topology of G^G. We shall put P=the product topology of G^G and S=the S-topology on $C(G, G)$. Then, if G is a T_1-space, we can easily see that the relative topology on $C(G, G)$ from P is coarser than the S-topology on $C(G, G)$. That is,

$$\mathfrak{U} = P \subset S_T \text{ on } G.$$

PROPOSITION 1.1. *If S' is the family of all closed subsets with nonempty interiors, and if S' denotes the S-topology, then following hold.*

(i) $S'_T \subset S_T$

(ii) *If G is a regular space then* $S'_T \supset P = \mathfrak{U}$

(iii) *If G is a normal space then* $S_T = S'_T \supset P = \mathfrak{U}$.

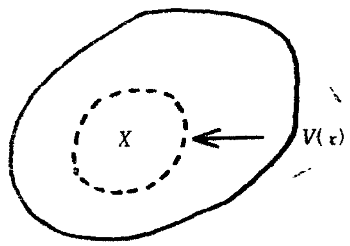
Proof. The subbase of the S'-topology on $C(G, G)$ is the family $\{T(M, U)\}$, where M runs over S' and U runs over $\mathfrak{U}$. Since $S' \subset S$ $S'_T \subset S_T$ is obvious.

Next, let G be a regular space. Then G is a T_1-space, and thus each point of G is a closed subset of G. The family $\{T(x, U) = \{r_g | g\epsilon G,\ r_g(x) = xg\epsilon U\}\}$ is the subbase of the topology $P = \mathfrak{U}$ on G, where $U \epsilon \mathfrak{U}$. Since $x \epsilon S$ $T_r(x, U) \epsilon S_T$, which means that $P \subset S_T$. Since G is regular, for each $x \epsilon G$ and an open neighborhood $V(x)$ there exists an open neighborhood $U(x)$ of x such that $\overline{U(x)} \subset V(x)$, where $\overline{U(x)}$ is the closure of $U(x)$. In this case $\overline{V(x)} \epsilon S'$. Since $x \epsilon \bar{V}(x)$

$T_r(\overline{V(x)}, U)$ $(\epsilon S'_T$ on $G) \subset T_r(x, U)$ $(\epsilon P$ on $G)$, where $T_r(\overline{V(x)}, U) = \{r_g | g\epsilon G,\ r_g(\overline{V(x)} \subset U\}$. Since

$$\bigcap_{x \epsilon V(x)} \overline{V(x)} = x,$$

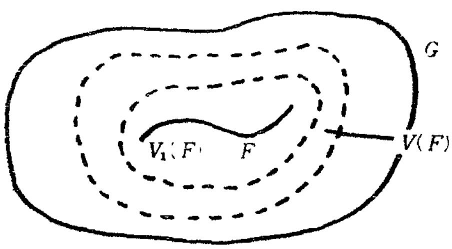
bacause of that as in the diagram



$G-V(x)$ (a closed subset of G) and x are separated by two disjoint open sets, that is, there exists an open subset $V_1(x)$ such that $x \in V_1(x) \subset V(x)$, we have

$$\bigcup_{x \in V(x)} T_r(V(x), U) = T_r(x, U).$$

Therefore $T_r(x, U)$ ($\in P$ on G) is an open subset in the topology S'_T on G, and thus $P \in S'_T$. Let G be a normal space. By (i) and (ii) it suffices to prove that $S_T \subset S'_T$ for (iii). Let a closed subset F of G have no any interior points. As in the diagram



we take an open neighborhood $V(F)$ of F, then $G-V(F)$ (a closed subset of G, and $F \cap (G-V(F)) = \phi$) and F are separated by two disjoint open subsets. Thus there is an open subset $V_1(F)$ such that $F \subset V_1(F) \subset V(F)$. In this case $\overline{V(F)}$ and $\overline{V_1(F)}$ are in S. Since

$$\bigcap_{F \subset V(F)} \overline{V(F)} = F$$

we have $\bigcap_{F\subset V(F)} T_r(\overline{V(F)},\ U)=T_r(F,U)$,

where $T_r(F,U)=\{r_g \mid g\in G,\ \ r_g(F)\subset U\}$. Therefore we have $S_T\subset S'_T$. q. e. d.

§2. Dual Groups

DEFINITION 2.1. Let G and $\boldsymbol{R}$ be an abelian (additive) topological group and the topological group consisting of all real numbers, respectively. For the set $\boldsymbol{Z}$ of all integers $\boldsymbol{R}/\boldsymbol{Z}$ is the *one-dimensional torus.* If $\delta : G \rightarrow \boldsymbol{R}/\boldsymbol{Z}$ is a continuous homomorphism, then δ is called a *character* of G. The set of all characters of G is called the dual group of G, written G'.

For σ_1 and $\sigma_2\epsilon G$ and $x\epsilon G$ we define

$$(\sigma_1+\sigma_2)(x)=\sigma_1(x)+\sigma_2(x),$$

then with this addition G' is an abelian (additive) group. Furthermore, by S-topology (§1) G' becomes a topological group. (Note that if G is a Hausdorff space then the S-topology is the compact open topology i. e, the k-topology). In particular, if G is a Hausdorff space then G' is also a Hausdorff space.

DEFINITION 2.2. A subset U of a group G is said to be *symmetric* if $U=U^{-1}$. In this case G is an abelian group, U is symmetric if $U=-U$.

LEMMA 2.3. *Let e be the identity of a topological group G. Then for each open neighborhood W of e there exists a symmetric open neighborhood U of e such that*

$$U^{\varepsilon_1}\cdots\cdots U^{\varepsilon_n}\subset W,$$

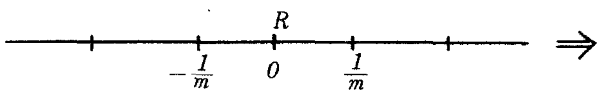
where $\varepsilon_i=\pm1$ for $i=1,2,\cdots\cdots n$.

Proof. At first, we shall prove the existence of U. Let $\{V\}$ be a fundamental system of open neighborhoods of e. For each $V\epsilon\ \{V\}$, since G is a topological group V^{-1} is also an open neighborhood of e. Put $U=V\cap V^{-1}$, then U is a symmetric open neighborhood of e, since $U^{-1}=(V\cap V^{-1})=V^{-1}\cap(V^{-1})^{-1}=V^{-1}\cap V$. Thus each V contains a symmetric open neighborhood of e. This means that there is a fundamental system $\{U\}$ of symmetric neighborhood of e.

In a topological group the mapping $G\times G\rightarrow G((x,y)\rightarrow xy^{-1})$ is continuous. Therefore, for each open subset W containing e there exists a symmetric open neighborhood U_1 such that $U_1U_1^{-1}\subset W$. Put $W_1=U_1\ U_1^{-1}$, which is an open subset containing e. Then, by the same reason as above there exists a symmetric open neighborhood U_2 of e such that $U_2U_2^{-1}\subset W_1$. Repeating this way, we get a symmetric open neighborhood U such that $U^{\varepsilon_1}\cdots\cdots U^{\varepsilon_n}\subset W$. q. e. d.

LEMMA 2. 4. ***R* and *R/Z* are locally homeomorphic.**

Proof. For each positive integer m, $-\frac{1}{m}$ is $\frac{m-1}{m}$ in R/Z.
As in the diagram



an open neighborhood $(-\frac{1}{m}, \frac{1}{m})$ of O in R is homeomorphic to the open neighborhood $(\frac{m-1}{m}, \frac{1}{m})$ (see the above diagram). Thus R and R/Z are locally homeomorphic. q. e. d.

DEFINITION 2. 5. (Equicontinuous set) Let E and F be two topological spaces. A family L of functions in $C(E, F)\subset F^E$ where F is a ***uniform space***, is called an ***equicontinuous set*** at $x\in E$ if for each element V in the ***uniformity*** of F there exists an open neighborhood U of x such that $(f(y), f(x))\in V$ for all $y\in U$ and $f\in L$ ([1]~[3]).

The P-closure (P=product topology) of an equicontinuous set in F^E is also equicontinuous. On each equicontinuous set in $C(E, F)$, the relative topology of k coincide with the relative topology of P. Above equicontinuous sets there is the Ascoli's theorem ([2]) such that if E is a Hausdorff locally compact topological space and F a Hausdorff uniform space, and a subset L of $C(E, F)$ with the k-topology then L is k-compact if and only if

(a) G is k-closed,

(b) for each $x \in E$ the closure of the set $\{f(x) \mid f \in L\}$ is compact in F,

(c) L is equicontinuous.

THEOREM 2.6. *If G is a locally compact Hausdorff abelian topological group, then its dual group G' is also a locally compact Hausdorff abelian topological group.*

Proof. Since the dual group G' is a Hausdorff abelian topological group, we have to prove that G' is locally compact. To do this, we take U which is a neighborhood of O of G such that $\bar{U}$ is compact. Let V_m be an open neighborhood of O in $\boldsymbol{R}/\boldsymbol{Z}$ such that $V_m = \{x \in \boldsymbol{R}/\boldsymbol{Z} \mid -\frac{1}{m} < x < \frac{1}{m}$, m is a positive integer$\}$ (see Lemma 2.4.), and let

$$(\boldsymbol{R}/\boldsymbol{Z})_m = \{x' \in G' \mid x'(x) \in V_m \text{ for } x \in \bar{U}\}.$$

Then $(\boldsymbol{R}/\boldsymbol{Z})_m$ is an open neighborhood of $\boldsymbol{O}'$ of G' in the k-topology. We want to prove that $(\boldsymbol{R}/\boldsymbol{Z})_m$ is compact in the k-topology. Using Lemma 2.3, we can prove that there is an open neighborhood U_1 of O of G such that for all $x' \in (\boldsymbol{R}/\boldsymbol{Z})_m$ and $x, y \in U_1$, $x-y \in U_1$ implies

$$|x'(x-y)| = |x'(x) - x'(y)| < \varepsilon.$$

This means that $(\boldsymbol{R}/\boldsymbol{Z})_m$ is equicontinuous (Definition 2.5.). By the above remark, $(\boldsymbol{R}/\boldsymbol{Z})_m$ in the P-topology is also equicontinuous. Therefore $(\boldsymbol{R}/\boldsymbol{Z})_m$ in the k-topology is also equicontinuous. Note that $\boldsymbol{R}/\boldsymbol{Z}$ is a uniform space. Since $\boldsymbol{R}/\boldsymbol{Z}$ is compact, for each $x \in G$ and any subset H of G, $H(x) = \{f(x) \mid f \in H\}$ is relatively compact in $\boldsymbol{R}/\boldsymbol{Z}$. Thus, by the Ascoli's Theorem $(\overline{\boldsymbol{R}/\boldsymbol{Z}})_m$ in the k-topology is k-compact, and hence the identity $\boldsymbol{O}'$ of G' has a compact k-open neighborhood $(\boldsymbol{R}/\boldsymbol{Z})_m$. Since G' is a topological group, it is a locally compact space. q. e. d.

§3. Examples

By Theorem 2.6, if G is a Hausdorff locally compact abelian topological group then so is its dual group G'.

THEOREM 3.1. *Under the above situation*

(i) *if G is compact then G' is discrete.*

(ii) *if G is discrete then G' is compact*

(Note that a discrete space is locally compact).

Proof. For (i) suppose that G is compact. We want to prove that there exists an open neighborhood V of O of $\boldsymbol{R}/\boldsymbol{Z}$ such that $T(G, V)=\{O'\}$. Let us put $V=\{x\epsilon\boldsymbol{R}/\boldsymbol{Z} \mid -\frac{1}{m}<x<\frac{1}{m}$, m is a positive integer$\}$ (see Lemma 2.4). Assume that $T(G, V)\neq\{O'\}$, then there is an element $x'\epsilon T(G, V)$ such that for some $x\epsilon G$ $x'(x)\neq 0$. Then by Lemma 2.3, there exists an open neighborhood V_1 of O of $\boldsymbol{R}/\boldsymbol{Z}$ such that

$$\sum_{i=1}^{n} V_i \subset V, \quad V_i=V_1 \text{ for } i=1, 2, \cdots\cdots n,$$

and $nx'(x) > \frac{1}{m}$ for a large positive integer n. But then we have $nx'(x)=x'(nx)\epsilon V$ (see Lemma 2.3).

This is contrary to the choice of n and V_1, and thus $x'(x)=O$ i.e., $x'=O'$.

For (ii), assume that G is discrete. Then $\{O\}$ is a compact open neighborhood of O of G. Thus for each neighborhood V of O of $\boldsymbol{R}/\boldsymbol{Z}$ and for each $x'\epsilon G'x$ $(O)=O \epsilon V$. This means that $\overline{T(\{O\},\ V)}=\overline{G'}=G$ is compact (see theorem 2.6). Thus the proof of (ii) is completed.

EXAMPLE 3.2. The dual group $\boldsymbol{R}$ of $\boldsymbol{R}$ is homeomorphic to $\boldsymbol{R}$.

Proof. Recall that $\boldsymbol{R}$ is a one-dimensional vector space over itself. Each linear transformation $f:\boldsymbol{R}\rightarrow\boldsymbol{R}$ is a continuous homomorphism. Let $p:\boldsymbol{R}\rightarrow\boldsymbol{R}/\boldsymbol{Z}$ be the canonical projection. Then p is a continuous homomorphism. For a linear transformation f: $\boldsymbol{R}\rightarrow\boldsymbol{R}$ $pf:\boldsymbol{R}\rightarrow\boldsymbol{R}/\boldsymbol{Z}$ is a character of $\boldsymbol{R}$. It is easily proved that each character of $\boldsymbol{R}$ has the form pf for some linear transformation.

Let $M(\boldsymbol{R})$ be the set of all linear transformations from $\boldsymbol{R}$ to $\boldsymbol{R}$. Then the topological space $M(\boldsymbol{R})$ with k-topology is homeomorphic to $\boldsymbol{R}$.

If for two linear transformations $f\neq g\epsilon M(\boldsymbol{R})$ then $pf\neq pg$. Hence $\boldsymbol{R}'\cong M(\boldsymbol{R})$ as topological groups. Hence $\boldsymbol{R}'\cong\boldsymbol{R}$ and $\boldsymbol{R}'$ is also a Hausdorff locally compact abelian topological group.

EXAMPLE 3.3. Let $\boldsymbol{R}$ be a discrete abelian topological group consisting of all real numbers. Then $\boldsymbol{R}'$ is equal to $M(\boldsymbol{R})$ as sets. But with the k-topology given on $\boldsymbol{R}'$ it is compact.

EXAMPLE 3.4. Let $\boldsymbol{Z}$ be a discrete abelian topological group consisting of all integers. Then $\boldsymbol{Z}$ is homeomorphic to S', where S' is the one-dimensional space.

Proof. For each linear transformation $f\in M(\boldsymbol{R})$ $f/\boldsymbol{Z}$ is a continuous homomorphism from $\boldsymbol{Z}$ to $\boldsymbol{R}/\boldsymbol{Z}$. In fact, f is a some real number $\sigma_f\in\boldsymbol{R}$, and for $m\in\boldsymbol{Z}$ $f(m)=\sigma_f m$. mod. $\boldsymbol{Z}$. For $\frac{1}{n}$ and $m+\frac{1}{n}$ and $e\in\boldsymbol{Z}$

$$\frac{e}{n}\equiv(m+\frac{1}{n})e \text{ mod. } \boldsymbol{Z}.$$

Therefore we have

$$\boldsymbol{Z}'\cong S'\cong R/\boldsymbol{Z}$$

as topological groups. Of course $\boldsymbol{Z}'$ is compact. q. e. d.

References

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