Bull. Korean Math. Soc.
Vol. 13, No. 1, 1976.

THE STONE-CECH COMPACTIFICATION OF A LOCALLY COMPACT SEPARABLE LINDELÖF SPACE

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Let X be a pseudocompact normal Hausdorff space and let U be a C^*-embedded dense subset of X, Assuming the continuum hypothesis, it is shown that $X=\beta U$ if U is locally compact, separable and Lindelöf.

In this paper, all spaces are completely regular and Hausdorff. We assume the validity of the continuum hypothesis to the effect that every set with cardinality c can be made into a well ordered set in which initial segments are countable sets. This point is essential in proving the following result, which may serve as an alternate definition of the Stone-Cech compactification βU for U satisfying the conditions listed in the title of this paper.

THEOREM (CH) *Let X be a pseudocompact normal space and Let U be a C^*-embedded dense subset of X. If U is locally compact, separable and Lindelöf, then X is compact.*

Here and in what follows, (CH) means that the result is true under the continuum hypothesis. The letters U and N will always be used to denote the space U of the above theorem and the discrete space of integers, respectively.

LEMMA 1. *A space X with $U\subset X\subset\beta U$ is pseudocompact if and only if it meets every nonempty zero set of $\beta U-U$.*

LEMMA 2. *Every Lindelöf subspace Y of $\beta U-U$ is C^*-embedded in βU.*

These are trivial results. For the first, observe that there is an unbounded $f\in C(X)$ exactly when X is disjoint from some zero set of $\beta U=\beta X$ on which the Stone extension of f assumes the value ∞. The second follows from the Tietze theorem since Y must be closed in the Lindelöf space $U\cup Y$.

LEMMA 3. (CH) *If p is a point of $\beta U-U$, there is a collection $\{Z_\alpha\}$ of* *zero sets of βU-U containing p indexed by all countable ordinals α such that* (1) $Z_\alpha \subset Z_\gamma$ *if* $\alpha > \gamma$, *and* (2) *for any neighborhood V of p, there is an α with $Z_\alpha \subset V$.*

Proof. Since βU-U is a C^*-embedded zero set of βU as U is locally compact and Lindelöf, separability of U implies that there are at most c zero sets of βU-U containing p. But p is not a G_δ-point [1, Corollary 9.6], and there are exactly c such zero sets. Then, by the continuum hypothesis, they can be put in one to one correspondence with all countable ordinals. For our purpose, it is enough to let Z_α be the intersection of those zero sets corresponding to the ordinals ν with $\nu \leq \alpha$.

The following is a sharpened version of a result of Hindmann [2, Theorem 2.2].

LEMMA 4. (CH) *If p is a point of $\beta N-N$, there is a collection $\boldsymbol{V}$ of pairwise disjoint open subsets of $\beta N-N$ with $|\boldsymbol{V}|=c$ such that if V, W are distinct members of $\boldsymbol{V}$, then p is the only limit point shared by V and W.*

Proof. By Lemma 3, there is a nested collection $\{Z_\alpha\}$ of zero sets of $\beta N-N$ indexed by countable ordinals such that (1) if $\alpha > \gamma$ then $p \in Z_\alpha \subset Z_\gamma$, and (2) for each neigborhood V of p, there is an α with $Z_\alpha \subset V$. We choose by transfinite construction pairwise disjoint cozerosets V_α of $\beta N-N$ as follows: Let α be a countable ordinal and suppose that we have already chosen pairwise disjoint nonempty cozero sets V_γ with $V_\gamma \subset Z_\gamma - \{p\}$ for all γ less than α. Deleting these cozero sets from Z_α, we obtain a zero set Z_α' containing p. Since Z_α' has nonempty interior [1, 6S8], we can choose a cozero set V_α contained in $Z_\alpha - \{p\}$ disjoint from all V_γ, $\gamma < \alpha$. By [1, 6Q], each V_α contains nonempty cozero sets $V_{\alpha t}$ of $\beta N-N$ with t running over the real numbers such that $V_{\alpha s}$ and $V_{\alpha t}$ are disjoint whenever $s \neq t$. For each t let V_t denote the union of all $V_{\alpha t}$ and let $\boldsymbol{V}$ denote the collection of disjoint open sets V_t. By construction, p is a limit point of every V_t. If V_s and V_t, $s \neq t$, had a common limit point other then p, it must be a common limit point of $V_s - Z$ and $V_t - Z$ for some zero set neighborhood Z of p. Accordingly, to complete the proof, it suffices to prove that $V_s - Z$ and $V_t - Z$ have disjoint closures in βN for every zero set neighborhood Z of p. But, since every neighborhood of p must contain all but countably many $V_{\alpha s}$ and $V_{\alpha t}$, both $V_s - Z$ and $V_t - Z$ are cozero sets of $\beta N - N$. Hence, $V_s - Z$ and $V_t - Z$ have disjoint closures by Lemma 2, completing the proof of Lemma 4.

COROLLARY. (CH) *Let X be a normal space with $N \subset X \subset \beta N$. If $X - N$ is dense in $\beta N - N$, then $X = \beta N$.*

In other words, a pseudocompact dense subspace of βN is βN if it is normal.

Proof. If there is a point p in $\beta N - X$, choose by Lemma 4 disjoint open subsets V and W of $\beta N - N$ such that p is the only limit point shared by V and W. Since $X - N$ is dense in $\beta N - N$, the sets $V \cap X$ and $W \cap X$ can not be completely separated although they have disjoint closures in X. This proves the first statemnst of the lemma. The second statement follows from Lemma 1 since every nonempty zere set of $\beta N - N$ contains a nonempty open subset of $\beta N - N$ by [1, 6S].

With these preparations, we proceed to prove the main result.

Proof of Theorem. Since U is C^*-embedded and dense in X, we have $\beta X = \beta U$. If there is a point p in $\beta U - X$, let $\{Z_\alpha\}$ be a collection of zero sets of $\beta U - U$ with properties described in Lemma 3. Using induction, we construct a monotone collection of discrete subsets D_α of $X - U$ as follows. Let α be a countable ordinal and suppose that we have already chosen zero sets Z_γ' of βX-U with $p \in Z_\gamma' \subset Z_\gamma$ and points q_γ of X from $Z_{\gamma\gamma}' - \{p\}$, $\gamma < \alpha$, in such a way that (1) p is not a limit point of $D_\gamma = \{q_\eta | \eta \leq \gamma\}$, $\gamma < \alpha$, (2) $\beta < \gamma$ implies $Z_\beta' \supset Z_\gamma'$ and (3) Z_γ' is completely separated from the union of all D_β, $\beta < \gamma$. It is clear that the sets D_γ, $\gamma < \alpha$, as well as their union $\cup D_\gamma$ are discrete and countable. Since the pseudocompact space X is countably compact by normality, $\cup D_\gamma$ must have a countably compact normal closure in X. But the countable discrete space $\cup D_\gamma$ is C^*-embedded in X by Lemma 2, and it must have compact closure in X by the Corollary to Lemma 4. Therefore, p is not a limit point of $\cup D_\gamma$. Accordingly, Z_α contains a zero set of βX-U containing p which is completely separated from $\cup D_\gamma$. Let Z_α' be the intersection of this zero set with all Z_γ', $\gamma < \alpha$, and let q_α be a point of X lying in Z_α'-$\{p\}$. It follows that The set $D_\alpha = \{q_\gamma \mid \gamma \leq \alpha\}$ is also a discrete set failing to have p as limit point. This completes our transfinite construction and we have the proposed sets D_α. Let E and F be uncountable disjoint subsets of $\cup D_\alpha$. Since any uncountable set is confinal in the set of all countable ordinals, p is a limit point of E and F. To see that E and F do not have any common limit point other than p, let W be a neighborhood of p. Since W contains all but a countable number of points q_α, the disjoint discrete sets $E-W$ and $F-W$ are both countable. Hence, $E-W$ and $F-W$ have disjoint closures in $\beta X=\beta U$ by Lemma 2. We have shown that p is the only limit point shared by E and F. That is, the subsets E and F of X have disjoint closures in X but not in βX. Since this contradicts to the normality of X, it follows that $X=\beta X$, completing the the proof. ∎

Reforences

[1] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, 1960.

[2] Neil Hindman, *On the existence of c-points of* $\beta N \setminus N$, Proc. Amer. Math. Soc., **21**(1969), 277–280.

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