Bull. Korean Math. Soc.
Vol. 13, No. 1, 1976.

PL INVOLUTIONS OF 3-MANIFOLDS

BY KYUNG WHAN KWUN*

1. Introduction

My general purpose here is to present recent developments in the theory of *PL* involutions of 3-manifolds. Later, I will point out drastic deviations in high dimensional case. The presentation will be necessarily limited and incomplete but I will try to make it coherent.

2. *PL* involutions of the 3-sphere and 3-space

2.0. If h is an involution of a space X, the fixed point set of h will be denoted by $F(X, h)$ or simply by $F(h)$. Suppose h is a *PL* involution of a finite complex K. Then it is not difficult to show that h is simplicial on some subdivision of K. If h is a *PL* involution of a 3-manifold M, likewise there exists a triangulation of M with respect to which h is simplicial.

2.1. Now let h be a *PL* involution of S^3. If $F(h)$ is a 2-sphere, it is easy to see that h is equivalent to the reflection of S^3 through its equator. It is known [30] that $F(h)$ is an r-sphere, $r=-1, 0, 1$, or 2. Case $r=-1, 0, 1$ had been unsolved until Livesay [24, 25] solved cases $r=-1$ and 0 in 1963. In each of the cases $r=-1$ and 0, h is equivalent to the standard one. The remaining case $r=1$ was finally solved by Waldhausen [37] in 1969.

2.2. Now let h be a *PL* involution of 3-space. The reason why this case is not a corollary to the case of S^3 is that the one point compactification of h need not be *PL* even though h is. According to Smith [30], of course $F(h)$ is an r-space, $r=0, 1$ or 2. The case $r=0$ was solved by Livesay. The case $r=2$ was solved by Harrold and Moise [4] who showed that if a 2-sphere in S^3 is locally tame except at a point, then at least one complementary domain has a 3-cell as closure. The case $r=1$ was finally solved by Kwun and Tollefson [20] in 1973 who in fact show that if X is a closed 3-manifold

(*) Supported in part by NSF Grants GP 38602 and MPS 73-03747 A02.

and h is an involution of X such that $h|X-a$ is equivalent to a PL involution, where $a \in F(h)$, then h itself is equivalent to a PL one. Thus there are exactly three non-equivalent PL involutions of 3-space.

3. *PL* involutions of 3-manifolds

PL involutions of 3-manifolds in general had not been studied in depth until recently mainly because there weren't enough tools or techniques. By this time, we have a proof of Dehn's lemma [28], sphere theorem [38], the fibering theorem [31] of Stallings, the product theorem [3] of E. Brown, the uniqueness of connected sum [26], and Waldhausen theory [36].

Using results like these and/or others. PL involutions of many individual 3-manifolds have been studied. For example, see [5, 6, 7, 8, 10, 27, 29, 13, 14, 15, 16, 32, 33, 34].

More recently, Tollefson [35] introduced a new technique which was refined and extended by himself and others, brought a new depth in analysing PL involutions of 3-manifolds. I particularly mention the following two results.

PRODUCT THEOREM (Kim and Tollefson [11]) *Let S be a compact connected and h a PL involution of $S\times[0,1]$ such that $h(S\times\{0,1\})=S\times\{0,1\}$. Then h is equivalent to some $\alpha\times\beta$, where α is an involution or the identity of S and $\beta(t)=1-t$ or t.*

EXTENTION THEOREM (Kwun and Tollefson [22]) *Let X be a compact 3-manifold and h a PL involution of the interior $\mathring{X}$ of X. Then there exists one and only one equivalent class of PL involutions of X whose restriciton to X is equivalent to h.*

The first theorem is clearly useful in studying PL involutions of $S\times[0,1]$ and $S\times S^1$, for example, [11, 21, 22]. The second theorem is also very useful. Until the extension theorem, PL involutions of such simple space as $\mathbf{R}^2\times S^1$ could not be classified. Now we know there exists exactly seven non-equivalent PL involutions. (Also see the next sections.)

4. Tame fixed point sets

Now that PL involutions of many 3-manifolds can be completely analysed, it would be nice to know when a given involution of a 3-manifold is equivalent to a *PL* one. This is answered by Kwun [19] for closed 3-manifolds. The necessary and sufficient condition is that $F(h)$ be tame. This result was just recently extended to a quite general case by Kwun and Tollefson [23]. The last result implies that (1) if A is the set of points where $F(h)$ fails to be locally tame, then A has no isolated point and (2) an involution h of a 3-manifold (with or without boundary, compact or not) is equivalent to a *PL* one if and only if $F(h)$ is tame.

Then it follows that Bing's examples [1, 2] of bad involutions of S^3 are simplest possible.

5. Involutions of high dimensional manifolds

Involutions of high dimensional manifolds behave quite differently. For example, for a closed n-manifold M, $n \geq 5$, there are infinitely many non-equivalent *PL* involutions h of $M \times [0, 1]$ such that $h(M \times 0) = M \times 0$ and no h is equivalent to a product involution. (See [18]). Also using [18, 19]. it was shown [22] that the extension theorem in Section 3 also fails in two ways. Some *PL* involution h of $\mathring{X}$ (for a suitable h and X) is not equivalent to the restriction of any *PL* involution of X and also for suitable h and X, it is possible to find infinitely many non-equivalent *PL* involutions $h_1, h_2, \ldots$ of X such that $h_i | \mathring{X}$ are all equivalent to h.

Regarding Section 4, it is easy to have an isolated bad point in $F(h)$ in high dimensions.

6. Final remarks

There has been a question whether every closed 3-manifold admits an involution. Tollefson first came up with a closed 3-manifold which admit no *PL* homeomorphism of a finite period. Just recently, Raymond and Tollefson showed that the same manifold has no homeomorphism of a finite period of any kind.

Kim and Tollefson found [12] a way to reduce involutions of many 3-manifolds to involutions of simpler 3-manifolds.

References

[1] R.H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned sphere*, Ann. of Math. **56** (1954), 354 - 362.

[2] ______, *Decompositions of E^3 into points and tame arcs*, Summer Institute on Set Theoretic Toplogy, Univ. of Wisconsin(1955), Revised 1958, 41-48.

[3] E.M. Brown, *Unknotting in $M^2\times I$*, Trans. Amer. Math. Soc. **203**(1966), 480-505.

[4] O.G. Harrold and E.E. Moise, *Almost locally polyhedral spheres*, Ann. of Math. **57**(1953), 575-578.

[5] J. Hempel, *Free cyclic actions on $S^1\times S^1\times S^1$*, Proc. Amer. Math. Soc. **40**(1975) 221-227.

[6] P.K. Kim, *PL involutions on lens spaces and other 3-manifolds*, Proc. Amer. Math. Soc. **44**(1974), 467-473.

[7] ______, *Cyclic actions on lens spaces*, to appear.

[8] ______, *Periodic homeomorphisms of the 3-sphere and related spaces*, Mich. Math. Jour. **21**(1974), 1-6.

[9] ______, *PL involutions on the non-orientable 2-sphere bundle over S^1*, to appear.

[10] ______, *Orientation reversing periodic PL maps of lens spaces*, to appear.

[11] P.K. Kim and J.L. Tollefson, *PL involutions of fibered 3-manifolds*, to appear.

[12] ______, *Splitting PL involutions on non-prime 3-manifolds*, to appear.

[13] K.W. Kwun, *3-manifolds which double-cover themselves*, Amer. Jour. Math. **91**(1969), 441-452.

[14] ______, *Piecewise linear involutions of $S^1\times S^2$*, Mich. Math. Jour. **16**(1969), 93-96.

[15] ______, *Searcity of orientation reversing PL involutions of lens spaces*, Mich. Math. Jour. **17**(1970), 305-308.

[16] K.W. Kwun, *Sense presrving PL involutions of some lens spaces*, Mich. Math. Jour. **20**(1973)73-77.

[17] ______, *Transfer homomorphisms of Whitehead groups of some cyclic groups*, Amer. Jour. Math. **93**(1973), 310-316.

[18] ______, *Transfer homomrphisms of Whitehead groups of some cyclic groups*, II, Proceedings of the Second Conference on Compact Transformation Groups, Springer-Verlag, 1972, 413-440

[19] ______, *An involution of a closed 3-manifold with a tame fixed point set is PL*, to appear.

[20] K. Kwun and J. Tollefson, *Uniqueness of orientation preserving PL involutions of 3-space*, Bull. Amer. Math. Soc. **79**(1973), 761-762.

[21] ______, *PL involutions of $S^1\times S^1\times S^1$*, Trans. Amer. Math.

Soc. **203**(1975), 97 - 106.

[22] ______, *Extending a PL involution of the interior of a compact manifold*, to appear.

[23] ______, *Involutions of 3-manifold with almost tame fixed point sets are PL*, to appear.

[24] G. R. Livesay, *Fixed point free involutions on the 3-sphere*, Ann. of Math. **72** (1960), 603 - 611.

[25] ______, *Involutions with two fixed points on the 3-sphere*, Ann. of Math. **78** (1963), 582 - 593.

[26] J. Milnor, *A unique decomposition theorem for 3-manifolds*, Amer. Jour. Math. **84**(1962), 1 - 7.

[27] M. Myung, *PL involutions of some 3-manifolds*, Proc. Amer. Math. Soc. **33** (1972), 576 - 581.

[28] C. Papakyriakopoulos, *Dehn's lemma and asphericity of knots*, Ann. of Math. **66**(1957), 1 - 26.

[29] G. X. Ritter, *Free actions of cyclic groups of order 2^n on $S^1 \times S^2$*, Proc. Amer. Math. Soc. **46**(1974), 137 - 140.

[30] P. A. Smith, *Transformations of finite period*, II, Ann. of Math. **40**(1939), 690 - 711.

[31] J. Stallings, *On fibering 3-manifolds*, *Topology of 3-manifolds and related topics* (Edited by M. K. Fort, Jr.), Prentice Hall, 1962, 95 - 100.

[32] Y. Tao, *On fixed point free involutions of $S^1 \times S^2$*, Osaka J. Math. **7**(1962), 145 - 152.

[33] J. Tollefson, *Free involutions on non-prime 3-manifolds*, Osaka J. Math. **7**(1970), 161 - 164.

[34] ______, *Homotopically trivial periodic homeomorphisms* on 3-manifolds, Ann. of Math. **96**(1973), 14 - 26.

[35] ______, *Involutions on $S^1 \times S^2$ and other 3-manifolds*. Trans. Amer. Math. Soc. **183**(1974), 139 - 152.

[36] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. **87**(1968), 56 - 88.

[37] ______ *Über Involutionen der 3-Sphäre*, Topology **8**(1969), 81 - 92.

[38] J. H. C. Whitehead, *On 2-spheres in 3-manifolds*, Bull. Amer. Math. Soc. **64** (1958), 161 - 166.

Michigan State University