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PROXIMITY SPACES AND COMPACTIFICATION

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1. Preliminary

The compactification problem for topological space which is completely regular go back to Stone (11) and Cech(12). Further development along this line are due to Smirnov (13): An account of this theory is to be found in the book of S. A. Naimpally and B. D. Warrack (7). But in this book, the Smirnov compactification of a separated proximity space was constructed by using clusters.

In this note, we try to construct the Smirnov compactification using ends which is the dual concept to that of clusters; the dual in the sense that two relations β and β^* on the power set of X are dual iff $A\beta^*B$ is equivalent to $\sim[A\beta(X-B)]$.

Efremovic's axioms of proximity relation δ are as follows;

(P1) $A\delta B$ implies $B\delta A$.

(P2) $(A\cup B)\delta C$ iff $A\delta C$ or $B\delta C$.

(P3) $A\delta B$ implies $A\neq\phi$, $B\neq\phi$.

(P4) $\sim(A\delta B)$ implies there exists a subset E such that $\sim(A\delta E)$ and $\sim[(X-E)\delta B]$

(P5) $A\cap B\neq\phi$ implies $A\delta B$.

A binary relation δ satisfying axioms (P1)−(P5) on the power set of X is called a (Efremovic) proximity on X. If δ also satisfies the following;

(P6) $x\delta y$ implies $x=y$,

then δ is called the separated proximity relation.

But it was known that there is an alternative approach to the study of proximity spaces by using the concept of δ-neighbourhood, the dual concept of δ.

Let $\ll$ be a binary relation on the power set of X satisfying the following conditions (N1)−(N6). Then B is said to be a δ-neighbourhood of A iff $A\ll B$.

(N1) $X \ll X$.

(N2) $A \ll B$ implies $A \subset B$.

(N3) $A \subset B \ll C \subset D$ implies $A \ll D$.

(N4) $A \ll B_i$ for $i=1, 2, \cdots, n$ iff $A \ll \bigcap_{i=1}^{n} B_i$.

(N5) $A \ll B$ implies $(X-B) \ll (X-A)$.

(N6) $A \ll B$ implies there is a subset C such that $A \ll C \ll B$.

Then it was also known that; a binary relation δ on the power set of X defined by $\sim(A\delta B)$ iff $A \ll (X-B)$, then δ is a proximity on X. Moreover, if $\ll$ also satisfies

(N7) $x \ll (X-y)$ iff $x \neq y$, then δ is separated.

The following results from the book of S. A. Naimpally and B. D. Warrack (7) are essential for our work.

LEMMA 1. *Let (X, δ) be a proximity space and let $\bar{A}$ and Int A denote, respectively, the closure and interior of A in $\mathscr{T}(\delta)$. Then*

(i) $A \ll B$ implies $\bar{A} \ll B$.

(ii) $A \ll B$ implies $A \ll$ Int B.

LEMMA 2. *If (X, δ) is a (separated) proximity space, then $\mathscr{T}(\delta)$ is (Tychonoff) completely regular.*

2. Ends, the dual concept to clusters

DEFINITION 1. An end α in a proximity space (X, δ) is a collection of subsets of X satisfying the following conditions;

(i) $B, C \in \alpha$ implies the existence of a non empty subset $A \in \alpha$ such that $A \ll B$ and $A \ll C$.

(ii) If $A \ll B$, then either $(X-A) \in \alpha$ or $B \in \alpha$

It was also known that α is an end iff the dual class $\alpha^* = \{E \subset X \mid (X-E) \notin \alpha\}$ of α is a cluster.

DEFINITION 2. A round filter (regular filter) $\mathcal{F}$ is a filter with the additional property that for each $F_1 \in \mathcal{F}$, there exists an $F_2 \in \mathcal{F}$ such that $F_2 \ll F_1$.

LEMMA 3. *α is an end iff it is a maximal round filter.*

LEMMA 4. *Every ultrafilter $\mathcal{F}$ in a proximity space contains a unique end $\mathcal{F}^0$.*

LEMMA 5. *$A \ll B$ in a proximity space (X, δ) iff every end in X contains either $X-A$ or B.*

The proofs of above Lemmas which are useful for our work may be found in the book of S. A. Naimpally and B. D. Warrack.

LEMMA 6. *Let α be an end in (X, δ). Then $A \in \alpha$ iff $\bar{A} \in \alpha$.*

proof If $A \in \alpha$, then there is a non-empty subset $A_2 \in \alpha$ such that $A_2 \ll A \subset \bar{A}$ which implies $A_2 \ll \bar{A}$. By Lemma 5, either $X-A_2 \in \alpha$ or $\bar{A} \in \alpha$. But the fact $A_2 \in \alpha$ excludes the first posibility, so that $\bar{A} \in \alpha$. Conversely, if $A \notin \alpha$, then $X-A \in \alpha$, so that there exists $A_2 \in \alpha$ such that $A_2 \ll X-A$ and by Lemma 1 (ii), $A_2 \ll Int(X-A) = X-\bar{A}$. By Lemma 5, either $X-A_2 \in \alpha$ or $X-\bar{A} \in \alpha$. Since $A_2 \in \alpha$ implies $X-A_2 \notin \alpha$, $X-\bar{A} \in \alpha$ or $\bar{A} \notin \alpha$.

LEMMA 7. *Let X be a non empty subset of a proximity space (Y, δ) and let α be an end in Y such that $X \in \alpha$. Then $\alpha' = \{A \cap X | A \in \alpha\}$ is an end in the subspace (X, δ_1).*

proof. By Lemma 3, α is a maximal (round) filter containing X. Then $\alpha' = \{A \cap X | A \in \alpha\}$, the trace of α on X, is a maximal filter in X. And if $A_1 \in \alpha$ then there exists $A_2 \in \alpha$ such that $A_2 \ll A_1$. Hence $(A_2 \cap X) \ll_1 (A_1 \cap X)$ which means α' is a round filter. Therefore α' is an end in X.

LEMMA 8. *For each $x \in X$, the collection*

$$\alpha_x = \{A | x \ll A\}$$

is an end. We call such an end a nbd-end and use the above notation.

proof. (i) $B, C \in \alpha_x \Longrightarrow x \ll B, \quad x \ll C$

$$\Longrightarrow \exists A_1, A_2 \subset X \text{ such that } x \ll A_1 \ll B, \ x \ll A_2 \ll C.$$

$$\Longrightarrow x \ll A_1 \cap A_2 \ll B, \ x \ll A_1 \cap A_2 \ll C.$$

Setting $A_1 \cap A_2 = A$, then A is also a member of α_x such that $A \ll B$, $A \ll C$.

(ii) $(X-A) \notin \alpha_x, B \notin \alpha_x \Longrightarrow \sim(x \ll X-A)$ and $\sim(x \ll B)$

$$\Longrightarrow x\delta A \text{ and } x\delta(X-B)$$

$$\Longrightarrow A\delta(X-B) \Leftrightarrow \sim(A \ll B).$$

LEMMA 9. *Let (X, δ) be a proximity space and let $x \in X$. Then $x \ll A$ iff A is a topological neighbourhood of x with respect to $\mathscr{T}(\delta)$.*

proof If $x \ll A$, then $x \ll Int\ A \subset A$, showing that A is a topological neighborhood. To prove the converse, let A be a topological open neighbourhood of x and assume $\sim(x \ll A)$, i.e. $x\delta(X-A)$. Then $x \in \overline{X-A} = X - IntA = X - A$, hence $x \notin A$. This is a contradiction. Therefore $x \ll A$. (Without loss of generality, we may work only with open neighbourhood A of x since if A is not, then there is an open set A_1 such that $x \in A_1 \subset A$ and we may work with A_1. Moreover $A_1 \subset A$, $x \ll A_1$ implies $x \ll A$.)

THEOREM 10. *Let α be an end in a proximity space (X, δ). Then α as a maximal filter converges to x iff $\alpha = \alpha_x$.*

proof. Assume $\alpha = \alpha_x$, then $x \ll F$ for every $F \in \alpha$. Lemma 9 implies that δ-neighbourhood and topological neighbourhood are coincident, hence α converges to x. Conversely, if α converges to x, then neighbourhood system α_x is a subclass of α, i.e. $\alpha_x \subset \alpha$. But α_x is the maximal, hence $\alpha_x = \alpha$.

THEOREM 11. *A proximity space is compact iff every end in the space is a nbd-end.*

proof This result follows from above Theorem and Lemma 4.

3. Smirnov compactification

We shall work exclusively with separated proximity space throughout this section.

Let $\mathfrak{X}$ be the family of all ends in X and let for $A \subset X$, $0(A) = \{\alpha \in \mathfrak{X} \mid A \in \alpha$ or α_x for $x \in A\}$. For $x \in X$, let $f(x) = \alpha_x$, then it is clear that

(1) f is a one to one mapping.

(2) $f(A) \subset 0(A)$.

It is to obtain property (1) that we insist on the proximity being separated, for we are then assured that $x \neq y$ implies $\alpha_x \neq \alpha_y$.

LEMMA 12. *Let A, B the subsets of X, then*

i) $0(A) \cup 0(B) \subset 0(A \cup B)$.

ii) $0(A) \cap 0(B) \subset 0(A \cap B)$.

proof.

i) $\alpha \in 0(A) \cup 0(B) \Rightarrow \alpha \in 0(A)$ *or* $\alpha \in 0(B)$

$\Rightarrow (A \in \alpha$ or α_x for $x \in A)$ or $(B \in \alpha$ or α_y for $y \in B)$

$\Rightarrow A\cup B\in\alpha$ or α_x for $x\in A\cup B$.

$\Rightarrow\alpha\in 0(A\cup B)$.

ii) $\alpha\in 0(A)\cap 0(B)\Rightarrow\alpha\in 0(A)$ and $\alpha\in 0(B)$

$\Rightarrow(A\in\alpha$ or α_x for $x\in A)$ and $(B\in\alpha$ or α_y for $y\in B)$

$\Rightarrow A\cap B\in\alpha$ or α_x for $x\in A\cap B$.

$\Rightarrow\alpha\in 0(A\cap B)$.

THEOREM 13. *Define* $\sim(\pi_1\delta^*\pi_2)$ *iff there exist* A *and* B *such that* $\sim(A\delta B)$, $\pi_1\subset 0(A)$, $\pi_2\subset 0(B)$. *Then* δ^* *is a separated proximity on* $\mathfrak{X}$.

proof. (P1) It follows directly from the symmetry of δ.

(P2) Assume $\sim[(\pi_1\cup\pi_2)\delta^*\pi_3]$. Then there exist A and B such that $\sim(A\delta B)$, $(\pi_1\cup\pi_2)\subset 0(A)$, $\pi_3\subset 0(B)$, hence $\pi_1\subset 0(A)$, $\pi_2\subset 0(A)$ and $\pi_3\subset 0(B)$, which implies $\sim(\pi_1\delta^*\pi_3)$ and $\sim(\pi_2\delta^*\pi_3)$. Conversly, if $\sim(\pi_1\delta^*\pi_3)$ and $\sim(\pi_2\delta^*\pi_3)$, then there exist A_1, B_1 such that $\sim(A_1\delta B_1)$, $\pi_1\subset 0(A_1)$, $\pi_3\subset 0(B_1)$ and A_2, B_2 such that $\sim(A_2, \delta B_2)$, $\pi_2\subset 0(A_2)$, $\pi_3\subset 0(B_2)$. Then $\pi_1\cup\pi_2\subset 0(A_1)\cup 0(A_2)\subset 0(A_1\cup A_2)$ and $\pi_3\subset 0(B_1)\cap 0(B_2)\subset 0(B_1\cap B_2)$ and obviousely $\sim[(A_1\cup A_2)\delta(B_1\cap B_2)]$. Thus $\sim[(\pi_1\cup\pi_2)\delta^*\pi_3]$.

(P3) Since $\sim(\phi\delta B)$ for each $B\subset X$ and $\phi\subset 0(\phi)=\phi$. $\sim(\phi\delta^*\pi)$ for every $\pi\subset\mathfrak{X}$.

(P4) $\sim(\pi_1\delta^*\pi_2)\Rightarrow\exists A, B$ such that $\sim(A\delta B)$, $\pi_1\subset 0(A)$ and $\pi_2\subset 0(B)$ $\Rightarrow\exists E\subset X$ such that $\sim(A\delta E)$, $\sim[(X-E)\delta B]$.
Setting $\pi_3=0(E)$, we have $\sim(\pi_1\delta^*\pi_3)$ since $\sim(A\delta E)$, $\pi_1\subset 0(A)$, $\pi_3\subset 0(E)$ and $\sim[(\mathfrak{X}-\pi_3)\delta^*\pi_2]$ since $\sim[(X-E)\delta B]$, $\mathfrak{X}-\pi_3=\mathfrak{X}-0(E)\subset 0(X-E)$, $\pi_2\subset 0(B)$.

(P5) $\sim(\pi_1\delta^*\pi_2)\Rightarrow\exists A, B\subset X$ such that $\sim(A\delta B)$, $\pi_1\subset 0(A)$, $\pi_2\subset 0(B)$
$\Rightarrow\exists A, B\subset X$ such that $A\cap B=\phi$, $\pi_1\subset 0(A)$, $\pi_2\subset 0(B)$
$\Rightarrow\pi_1\cap\pi_2=\phi$ since $0(A)\cap 0(B)=\phi$.

(P6) $\alpha_1\neq\alpha_2\Rightarrow\exists A\subset X$ such that $A\in\alpha_1$ but $A\notin\alpha_2$ and $\exists B\subset X$ such that $B\notin\alpha_1$ but $B\in\alpha_2$
$\Rightarrow X-B\in\alpha_1$ since α_1 is an ultrafilter and $B\notin\alpha_1$
$\Rightarrow\exists C\in\alpha_1$ such that $C\ll X-B$ or $\sim(C\delta B)$.

Then the fact $\sim(C\delta B)$, $\alpha_1\in 0(C)$ and $\alpha_2\in 0(B)$ implies $\sim(\alpha_1\delta^*\alpha_2)$

NOTATION Let $\mathscr{T}^*$ be the topology induced on $\mathfrak{X}$ by δ^*.

LEMMA 14. (X, δ) *is proximally isomorphic to* $f(X)$ *with the subspace proximity induced by* δ^* *and* $f(X)$ *is dense* in $\mathfrak{X}$.

proof. We first show that $f(X)$ is dense in $\mathfrak{X}$. Assume there is an $\alpha \in \mathfrak{X}$ such that $\sim[\alpha\delta^* f(X)]$, then there exist A and B such that $\sim(A\delta B)$, $\alpha \in 0(A)$ and $f(X) \subset 0(B)$. Then $f(X) \subset 0(B)$ implies $\alpha_x \in 0(B)$ for all $x \in X$, i. e. $x \ll B$ or $x \in B$ for all $x \in X$. This implies $B = X$, so that $A = \phi$ since $\sim(A\delta B)$ and $B = X$. Hence $\alpha \in 0(\phi) = \phi$ which is a contradiction.

Therefore $\alpha\delta^* f(X)$ for all $\alpha \in \mathfrak{X}$, i. e. $\overline{f(X)} = \mathfrak{X}$ or $f(X)$ is dense in $\mathfrak{X}$.

Now observe $\sim[f(A)\delta^* f(B)]$ iff $\sim(A\delta B)$ since $f(A) \subset 0(A)$ and $f(B) \subset 0(B)$. Thus X is proximally isomorphic to $f(X)$.

THEOREM 15. $(\mathfrak{X}, \delta^*)$ *is compact.*

proof. By theorm 10, it suffices to show that an arbitrary end $\mathfrak{A}$ in $\mathfrak{X}$ is a nbd-end. Since $f(X)$ is dense in $\mathfrak{X}$, Lemma 6 implies that $f(X) \in \mathfrak{A}$. From Lemma 7, there is a unique end $\mathfrak{A}'$ in $f(X)$ such that $\mathfrak{A}' \subset \mathfrak{A}$, namely $\mathfrak{A}' = \{\pi \cap f(x) \mid \pi \in \mathfrak{A}\}$. But X is proximally isomorphic to $f(X)$, so that there is an end α in X which corresponds to $\mathfrak{A}'$. To be specific, $\alpha = \{f^{-1}(\pi) \mid \pi \in \mathfrak{A}'\}$. We will now show that $\mathfrak{A} = \mathfrak{A}_\alpha = \{\pi \subset \mathfrak{X} \mid \alpha \ll^* \pi\}$. We first note that if $\pi \in \mathfrak{A}'$ then

$$f[X - f^{-1}(\pi)] = f\{f^{-1}[f(X)] - f^{-1}(\pi)\} = f\{f^{-1}[f(X) - \pi]\}$$
$$= f(X) - \pi \subset 0(X - f^{-1}(\pi)).$$

Now if $\pi \in \mathfrak{A}'$, then $\alpha \ll^* \pi$ as the following sequence of implications show;

$\pi \in \mathfrak{A}' \Rightarrow f^{-1}(\pi) \in \alpha$

$\Rightarrow$ there exists $\pi_1 \in \mathfrak{A}'$ such that $f^{-1}(\pi_1) \ll f^{-1}(\pi)$

$\Rightarrow \sim[f^{-1}(\pi)\delta X - f^{-1}(\pi)]$ and $\alpha \in 0(f^{-1}(\pi_1))$, $f(x) - \pi \subset 0(X - f^{-1}(\pi))$

$\Rightarrow \sim[\alpha\delta^* f(x) - \pi]$

$\Rightarrow \alpha \ll^* \pi$

Moreover if $\pi \in \mathfrak{A}$, then $\pi \cap f(X) = \pi_1 \in \mathfrak{A}'$, therefore $\alpha \ll^* \pi_1$ and $\pi_1 \subset \pi$, that is $\alpha \ll^* \pi$.

Combining the above sequence of Lemmas and Theorems, we obtain the main result of this note;

THEOREM 16. *Every separated proximity space* (X, δ) *is a dense subspace of a compact Hausdorff space* $\mathfrak{X}$.

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