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# DUALIZATION OPERATORS ON $P^2X$

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## I. Introduction.

The dual concepts of some mathematical notions are very important and they provide useful and powerful way for mathematician. In this note, we will study the dualization operators on the power set $PX$ of a set $X$ and $P^2X$ of a set $X$. We will also study the relations between these operators and investigate some examples and applications.

## II. Dualization operators.

Let $X$ be a set and $PX$ denote the power set of $X$. Define a map $d$ on $PX$ into $PX$ by $d(A)=X-A$. Let $id: PX \longrightarrow PX$ be the identity map. Then we obviously have;

$$d\circ d=id, \quad id\circ d=d\circ id=d \text{ and } id\circ id=id$$

We state this result formally;

LEMMA. *$(\{d, id\}, \circ)$, where $\circ$ is the operation of composition of maps, forms a group.*

Furthermore we have;

THEOREM 1. *Let $P^2X$ denote the power set of $PX$ of a set $X$. Define*

$$id: P^2X \longrightarrow P^2X \text{ by } id(\mathcal{A})=\mathcal{A}.$$
$$d_1: P^2X \longrightarrow P^2X \text{ by } d_1(\mathcal{A})=\{A\subset X \mid A\notin\mathcal{A}\}.$$
$$d_2: P^2X \longrightarrow P^2X \text{ by } d_2(\mathcal{A})=\{A\subset X \mid X-A\in\mathcal{A}).$$
$$d_3: P^2X \longrightarrow P^2X \text{ by } d_3(\mathcal{A})=\{A\subset X \mid X-A\notin\mathcal{A}\}.$$

*Then $(\{id, d_1, d_2, d_3\}, \circ)$ forms the Klein's four elements group, where the operation is given by the composition of maps.*

*Proof.* The verification of the following table of operation are straightforward from the definitions.

| $\circ$ | $id$ | $d_1$ | $d_2$ | $d_3$ |
|---|---|---|---|---|
| $id$ | $id$ | $d_1$ | $d_2$ | $d_3$ |
| $d_1$ | $d_1$ | $id$ | $d_3$ | $d_2$ |
| $d_2$ | $d_2$ | $d_3$ | $id$ | $d_1$ |
| $d_3$ | $d_3$ | $d_2$ | $d_1$ | $id$ |

Hence the result follows.

We shall call the maps in theorem 1 the dualization operators on $P^2X$. The following are somewhat simple examples to which dualization operators are applied.

EXAMPLE 1. Let $(X, \delta)$ be a proximity space, $\alpha \in P^2X$ an end in $X$ and $\alpha^* \in P^2X$ a cluster in $X$. Then $\alpha^* = d_3\alpha$.

*Proof* See [4], chapter 2(6. 11).

EXAMPLE 2. A filter $\mathcal{U}$ on a set $X$ is an ultrafilter iff $d_3\mathcal{U} = \mathcal{U}$.

*Proof.* Let $\mathcal{U}$ be an ultrafilter on $X$. Suppose that $A \in \mathcal{U}$. Then $X - A \notin \mathcal{U}$ since $A \cap (X - A) = \phi$. Thus $A \in d_3\mathcal{U}$, i. e., $\mathcal{U} \subset d_3\mathcal{U}$. Conversely, if $A \in d_3\mathcal{U}$ then $X - A \notin \mathcal{U}$. Since $X = (X - A) \cup A \in \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter, $A \in \mathcal{U}$. Thus $d_3\mathcal{U} \subset \mathcal{U}$ so that $\mathcal{U} = d_3\mathcal{U}$.

For sufficiency, let $\mathcal{U} = d_3\mathcal{U}$ and $\mathcal{U}$ be a filter. Then

$$\begin{aligned} A \cup B \in \mathcal{U} &\Longrightarrow A \cup B \in d_3\mathcal{U} \Longrightarrow X - (A \cup B) \notin \mathcal{U} \Longrightarrow (X - A) \cap (X - B) \notin \mathcal{U} \\ &\Longrightarrow X - A \notin \mathcal{U} = d_3\mathcal{U} \text{ or } X - B \notin \mathcal{U} = d_3\mathcal{U} \\ &\Longrightarrow A \in \mathcal{U} \text{ or } B \in \mathcal{U}. \end{aligned}$$

EXAMPLE 3. Let $X$ be a set and $\mathcal{A} \in P^2X$. Define

$$\text{sec } \mathcal{A} = \{B \subset X \mid A \cap B \neq \phi \text{ for each } A \in \mathcal{A}\} \in P^2X \text{ and}$$

$$\text{stack } \mathcal{A} = \{B \subset X \mid A \subset B \text{ for some } A \in \mathcal{A}\} \in P^2X.$$

Then 1) sec $\mathcal{A} = d_3(\text{stack } \mathcal{A})$ 2) stack $\mathcal{A} = d_3(\text{sec}\mathcal{A})$.

*Proof.* 1) $B \notin \sec \mathcal{A}$ iff $\exists\ A \in \mathcal{A}$ with $B \cap A = \phi$
iff $\exists A \in \mathcal{A}$ with $A \subset X - B$
iff $X - B \in \text{stack}\mathcal{A}$
iff $B \notin \{B \subset X \mid X - B \notin \text{stack}\mathcal{A}\}$.

2) $\sec \mathcal{A} = d_3(\text{stack}\mathcal{A}) \Longrightarrow d_3(\sec\mathcal{A}) = d_3{}^2(\text{stack}\mathcal{A}) = \text{stack}\mathcal{A}$.

## III. Application of dualization operators.

The theory of contiguity structures on a set $X$ has been introduced and studied by W. L. Terwilliger [5] and in an earlier, slightly different and more complicated form by V. M. Ivanova and A. A. Ivanov [2]. The concept of contiguity spaces is of central importance for the study of $T_1$-compactification of topological spaces. Terwilliger's axioms for contiguity structure on a set $X$ are as follows;

DEFINITION 1. Let $\varepsilon$ be a collection of finite subsets of $PX$ satisfying the following axioms;

C1) if a finite collection $\mathcal{A}$ corefines $\mathcal{B}$, i. e., for each $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ such that $B \subset A$, (in symbol, $\mathcal{A} < \mathcal{B}$) and $\mathcal{B} \in \varepsilon$ then $\mathcal{A} \in \varepsilon$.

C2) if $\mathcal{A}$ is finite collection with $\cap \mathcal{A} \neq \phi$ then $\mathcal{A} \in \varepsilon$.

C3) $\phi \neq \varepsilon \neq P^2X$.

C4) if $\mathcal{A}$ and $\mathcal{B}$ are finite collections with $\mathcal{A} \notin \varepsilon$ and $\mathcal{B} \notin \varepsilon$ then $\mathcal{A} \vee \mathcal{B} = \{A \cup B \mid A \in \mathcal{A},\ B \in \mathcal{B}\} \notin \varepsilon$.

C5) if $\mathcal{A}$ is a finite collection with $cl_\varepsilon \mathcal{A} = \{cl_\varepsilon A \mid A \in \mathcal{A}\} \in \varepsilon$ then $\mathcal{A} \in \varepsilon$. Here $x \in \mathrm{cl}_\varepsilon A$ iff $\{\{x\}, A\} \in \varepsilon$.

Then $\varepsilon$ is called a contiguity structure on $X$ and $(X, \varepsilon)$ a contiguity space.

We shall try to apply the dualization operators on the contiguity structure $\varepsilon$ and obtain some logically equivalent axioms.

PROPOSITION 2. *Let $\varepsilon'$ be a collection of finite subsets of $PX$ satisfying the following conditions;*

*$F$ 1) if $\mathcal{B}$ is a finite collection with $\mathcal{A} < \mathcal{B}$ and $\mathcal{A} \in \varepsilon'$ then $\mathcal{B} \in \varepsilon'$.*

*$F$ 2) if $\mathcal{A} \in \varepsilon'$ then $\cap \mathcal{A} = \phi$.*

*$F$ 3) $\phi \neq \varepsilon' \neq P^2X$.*

*$F$ 4) if $\mathcal{A} \in \varepsilon'$ and $\mathcal{B} \in \varepsilon'$ then $\mathcal{A} \vee \mathcal{B} \in \varepsilon'$.*

*$F$ 5) if $\mathcal{A} \in \varepsilon'$ then $\{\mathrm{cl}\ A \mid A \in \mathcal{A}\} \in \varepsilon'$, where $\mathrm{cl}\ A = \{x \in X \mid \{A, \{x\}\} \notin \varepsilon'\}$.*

*If we define $\mathcal{A} \notin \varepsilon'$ iff $\mathcal{A} \in \varepsilon$ then $\varepsilon$ is a contiguity structure on $X$.*

*Proof.* It is an immediate result of the definition.

Note that $\varepsilon' = d_1\varepsilon$ in the sense of theorem 1.

DEFINITION 2. Let $\varepsilon'$ be a collection of finite subsets of $PX$. Then $\varepsilon'$ is called a $c$-farness structure on $X$ if it satisfies the above conditions.

COROLLARY. *Let* $(X, \varepsilon)$ *be a contiguity space. Then* $\varepsilon' = d_1\varepsilon$ *is a c-farness structure on* $X$.

PROPOSITION 3. *Let* $\mu$ *be a collection of finite subsets of* $PX$ *satisfying the following;*

*U1) if* $\mathcal{A}\in\mu$ *and* $\mathcal{A}$ *refines a finite collection* $\mathcal{B}$ *(in symbol,* $\mathcal{A}\prec\mathcal{B}$*) then* $\mathcal{B}\in\mu$.

*U2) if* $\mathcal{A}\in\mu$ *then* $\cup\mathcal{A}=X$.

*U3)* $\phi\neq\mu\neq P^2X$.

*U4) if* $\mathcal{A}\in\mu$ *and* $\mathcal{B}\in\mu$ *then* $\mathcal{A}\wedge\mathcal{B}=\{A\cap B \mid A\in\mathcal{A}, B\in\mathcal{B}\}\in\mu$.

*U5) if* $\mathcal{A}\in\mu$ *then* $\{\mathrm{Int}_\mu A \mid A\in\mathcal{A}\}\in\mu$, where $Int_\mu A=\{x\in X \mid \{A, X-\{x\}\}\in\mu\}$. *If we define* $\mathcal{A}\in\mu$ *iff* $\{X-A \mid A\in\mathcal{A}\}\in\varepsilon'$ *then* $\varepsilon'$ *is a c-farness structure on* $X$.

*Proof.* We shall prove that U1)$-$U5) imply F1)$-$F5) respectively but it it obvious that U2) implies F2) and U3) implies F3).

First assume that $\mathcal{B}$ is a finite collection such that $\mathcal{A}<\mathcal{B}$ and $\mathcal{A}\in\varepsilon'$. Then $\{X-A \mid A\in\mathcal{A}\}\in\mu$ and $\{X-A \mid A\in\mathcal{A}\}\prec\{X-B \mid B\in\mathcal{B}\}$ and hence $\{X-B \mid B\in\mathcal{B}\}\in\mu$ or $\mathcal{B}\in\varepsilon'$.

Next assume that U4) holds, and let $\mathcal{A}$ and $\mathcal{B}$ be finite collectons with $\mathcal{A}\in\varepsilon'$ and $\mathcal{B}\in\varepsilon'$. Then $\{X-A \mid A\in\mathcal{A}\}\in\mu$ and $\{X-B \mid B\in\mathcal{B}\}\in\mu$. Hence

$$\{X-A \mid A\in\mathcal{A}\}\wedge\{X-B \mid B\in\mathcal{B}\}=\{(X-A)\cap(X-B) \mid A\in\mathcal{A}, B\in\mathcal{B}\}$$
$$=\{X-(A\cup B) \mid A\in\mathcal{A},\ B\in\mathcal{B}\}$$

belongs to $\mu$ by U4). Therefore $\mathcal{A}\vee\mathcal{B}\in\varepsilon'$.

Finally, assume that U5) holds and $\mathcal{A}\in\varepsilon'$. Then $\{X-A \mid A\in\mathcal{A}\}\in\mu$ and hence $\{Int_\mu(X-A) \mid A\in\mathcal{A}\}\in\mu$ by U5). Thus we have $\{X-Int_\mu(X-A) \mid A\in\mathcal{A}\}=\{cl_{\varepsilon'}A \mid A\in\mathcal{A}\}\in\varepsilon'$.

Note that $\mu=d_2\varepsilon'=d_2(d_1\varepsilon)=d_3\varepsilon$.

DEFINITION 3. Let $\varepsilon'$ be a collection of finite subsets of $PX$. Then $\mu$ is called a $c$-uniform covering structure on $X$ if it satisfies the above conditions.

COROLLARY. *Let $\varepsilon'$ be a c-farness structure on X. Then $\mu=d_2\varepsilon'$ is a c-uniform covering structure on X.*

*Proof.* The proof can be done dually.

We can also introduce a merotopic structure $\gamma$ which is logically equivalent with $\varepsilon, \varepsilon'$ and $\mu$ by defining $\gamma=\{\mathcal{A} | \mathcal{B} \cup \text{stack } \mathcal{A} \neq \phi$ for each $\mathcal{B} \in \mu\}$. The axioms for a $c$-merotopic structure $\gamma$ will be as follows;

Let $\mathcal{A}, \mathcal{B}$ be finite collections. Then $\gamma$ satisfies

S1) $\mathcal{A} < \mathcal{B}$, $\mathcal{A} \in \gamma$ implies $\mathcal{B} \in \gamma$.

S2) for all $x \in X$, $\{\{x\}\} \in \gamma$.

S3) $\phi \neq \gamma \neq P^2X$.

S4) $\mathcal{A} \vee \mathcal{B} \in \gamma$ implies $\mathcal{A} \in \gamma$ or $\mathcal{B} \in \gamma$.

S5) sec $\{clA | A \in \mathcal{A}\} \in \gamma$ implies sec $\mathcal{A} \in \gamma$, where $cl\, A=\{x | \text{sec}\{A, \{x\}\} \in \gamma\}$.

Actually $\gamma$ consists of all families of subsets of $X$ which contains arbitrarily small sets with respect to $\varepsilon(\mu)$.

In the study of contigual spaces, the contigual maps play so important roles as the continuous maps in topological spaces. The following proposition provides some alternative descriptions of contigual maps.

PROPOSITION 4. *If $f: (X, \varepsilon_1) \longrightarrow (Y, \varepsilon_2)$ is a map between contigual spaces, then the following are equivalent.*

*a) $\mathcal{A} \in \varepsilon_1$ implies $f(\mathcal{A}) \in \varepsilon_2$, i.e., $f$ is a contigual map.*

*b) $\mathcal{A} \in \varepsilon'_2$ implies $f^{-1}(\mathcal{A}) \in \varepsilon'_1$.*

*c) $\mathcal{A} \in \mu_{\varepsilon_2}$ implies $f^{-1}(\mathcal{A}) \in \mu_{\varepsilon_1}$.*

*Proof.* a)$\Longrightarrow$b) : Suppose $f^{-1}(\mathcal{A}) \notin \varepsilon'_1$, i.e., $f^{-1}(\mathcal{A}) \in \varepsilon_1$. Then $f(f^{-1}(\mathcal{A}) \in \varepsilon_2$ by a). Since $\mathcal{A} < f(f^{-1}(\mathcal{A}))$, we have $\mathcal{A} \in \varepsilon_2$ or $\mathcal{A} \notin \varepsilon'_2$.

b)$\Longrightarrow$a) : Suppose $f(\mathcal{A}) \notin \varepsilon_2$, i.e., $f(\mathcal{A}) \in \varepsilon'_2$. Then $f^{-1}(f(\mathcal{A})) \in \varepsilon'_1$ by b). Since $f^{-1}(f(\mathcal{A})) < \mathcal{A}$, we have $\mathcal{A} \in \varepsilon'_1$ by F1). Thus $\mathcal{A} \notin \varepsilon_1$.

b)$\Longrightarrow$c) : Let $\mathcal{A} \in \mu_{\varepsilon_2}$. Then $\{Y-A | A \in \mathcal{A}\} \in \varepsilon'_2$, and hence $\{f^{-1}(Y-A) | A \in \mathcal{A}\} = \{X-f^{-1}(A) | A \in \mathcal{A}\} \in \varepsilon'_1$ by b). Thus $\{f^{-1}(A) | A \in \mathcal{A}\} \in \mu_{\varepsilon_1}$,.

c)$\Longrightarrow$b) : Let $\mathcal{A} \in \varepsilon'_2$. Then $\{Y-A | A \in \mathcal{A}\} \in \mu_{\varepsilon_2}$, and hence $\{f^{-1}(Y-A) | A \in \mathcal{A}\} = \{X-f^{-1}(A) | A \in \mathcal{A}\} \in \mu_{\varepsilon_1}$ by c). Thus $\{f^{-1}(A) | A \in \mathcal{A}\} \in \varepsilon'_1$.

If we restrict our attention to the concept of nearness of two sets, that is,

$\delta_\varepsilon = \{\{A, B\} \mid A, B \subset X, \{A, B\} \in \varepsilon\}$, then we obtain the concept of Lodato proximity on $X$. Applying the dualization operators on $\delta\varepsilon$, we can also obtain corresponding equivalent axioms for Lodato proximity structure. But there seems to be no need to mention about those because it can be done similarly.

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