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ON FINSLER SPACES WITH ABSOLUTE PARALLELISM OF LINE-ELEMENTS

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§1. **Introduction.** This is a continuation of the previous paper [13]. In the paper [13], we combined the theory of M. Kurita [7], [8] with that of A. Deicke [3], [4] and suggested a method to study Finsler spaces. Now we fiix our eyes upon the following theorem [13]:

An n-dimensional Finsler space M with the absolute parallelism of line-elements in the sense of E. Cartan is realizable as an n-dimensional involutive distribution V on the figuratrix bundle N of M as follows:

(1) *The N is a $(2n-1)$-dimensional Riemannian manifold with the metric whose components are given by δ_{AB} with respect to an adapted orthogonal coframe.*

(2) *The D-connection defined on N becomes Riemannian.*

(3) *Any transformation between adapted orthogonal coframes is confined to a contact transformation (so called). A set of such transformations forms an orthogonal group, which is the fundamental group of N.*

(4) *The metric on N depends only on the local length and angular metric on M.*

(5) *The metric and connection on V induced from those on N are identified with the metric and connection on M respectively.*

This theorem suggests that it is possible to study a Finsler space M with the absolute parallelism of Cartan by the method of the Riemannian geometry only. The principal purpose of the present paper is to study the space M along the above statement. In the section 2, we choose a suitable adapted orthogonal coframe and give the equations of structure and the D-connection. In §3, we consider the distribution V and its complement $V^{\perp}$ and seek for the condition that they are parallel. We find, in §4, the conditions that the submanifolds corresponding to the two involutive distributions V and $V^{\perp}$ are totally geodesic, umbilic and minimal in N. In §5, We treat the osculating Riemannian space and give a condition for the space to be conformally flat.

In the final section, we consider Finsler spaces of constant curvature $K=0$, which are spaces with the absolute parallelism of line-elements in the sense of Cartan, and apply the results obtained in §4 to the spaces.

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§2 **Preleminaries.** Let M be an n-dimensional Finsler space with the fundamental

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function $F(x, y)$ $(y=\dot{x})$ and N be the figuratrix bundle on M. We denote by $g_{ij}(x, y)$ and $g^{ij}(x, y)$ the fundamental tensor $1/2(\partial^2F^2/\partial y^i\partial y^j)$ and its reciprocal one respectively and put

(2.1) $$y_i^*=g_{ij}y^i,\quad l_i^*=\partial F/\partial y^i,\quad l^i=y^i/F\ (=g^{ij}l_j^*).$$

Then, a local equation of N is given by $g^{ij}y_i^*y_j^*-1=0$ and the 1-form $\omega=l_i^*dx^i$ defines a contact structure on N except for the point (x, y^*) corresponding to (x, y) such that $F(x, y)=0$. In the sequel, we shall omit the marks * in y_i^* and l_i^* when no confusion occurs.

If we consider a matrix (ζ_i^a) of rank n satisfying

(2.2) $$g_{ij}=\sum_{a=1}^{n}\zeta_i^a\zeta_j^a,\qquad \zeta_i^n=l_i,\qquad \zeta_i^\alpha l^i=0\ (\alpha=1, 2, \cdots, n-1)$$

and denote by (ζ_a^i) the inverse of the matrix (ζ_i^a), we have

(2.3) $$g^{ij}=\sum_{a=1}^{n}\zeta_a^i\zeta_a^j,\qquad \zeta_n^i=l^i,\qquad \zeta_\alpha^i l_i=0,\qquad \zeta_a^i=g^{ij}\zeta_j^a.$$

From now on, we use indices as follows: Small Latin indices $a, b, c, \ldots\ldots$, $i, j, k, \ldots\ldots$ run from 1 to n and capital indices $A, B, C, \ldots\ldots$ from 1 to $2n-1$, while Greek indices $\alpha, \beta, \gamma, \ldots\ldots$ run from 1 to $n-1$.

Now we take $2n-1$ linearly independent 1-forms on N

(2.4) $$\omega^a=\zeta_i^a dx^i\ (\omega^n=\omega),\quad \omega_\alpha\overset{def}{=}\omega^{n+\alpha}=-\zeta_i^\alpha Dl^i,$$

where $Dl^i=dl^i+\Gamma^*{}_j{}^i{}_k l^j dx^k$ and $\Gamma^*{}_j{}^i{}_k$ are the connection coefficients of Cartan. Then, we can get a coframe (ω^A) on N composed of the members in (2.4), which is called an adapted orthogonal coframe. In this case, there exists an adapted orthogonal frame (e_A) such that $\omega^B(e_A)=\delta_A^B$. The equations of structure are given by

(2.5) $$d\omega^a=\omega^b\wedge\omega_b^a+\omega_\beta\wedge\mu^{\beta a},\quad d\omega_\alpha=\omega^a\wedge\nu_{a\alpha}+\sum_\beta\omega_\beta\wedge\omega_\beta^\alpha+(1/2)Z_{abc}\omega^b\wedge\omega^c,$$

together with

(2.6) $$\begin{gathered}\omega_b^a=\Gamma_b{}^a{}_c\omega^c+\Gamma_b{}^{a\gamma}\omega_\gamma,\quad \Gamma_b{}^a{}_c=-\zeta_{i\,|\,j}^a\zeta_b^i\zeta_c^j,\quad \Gamma_b{}^{a\gamma}=\zeta_i^a|_j\zeta_b^i\zeta_\gamma^j,\\ \mu^{\beta a}=-A_{ijk}\zeta_\beta^i\zeta_a^j\zeta_\gamma^k\omega^\gamma,\quad \nu_{a\alpha}=-A_j{}^i{}_{k\,|\,r}l^r\zeta_a^j\zeta_\alpha^k\zeta_i^\gamma\omega_\gamma.\\ Z_{abc}=R_{ijkh}l^i\zeta_a^j\zeta_b^k\zeta_c^h,\quad A_{ijk}=(1/2)F\partial g_{ij}/\partial y^k,\quad A_j{}^i{}_k=g^{is}A_{jsk},\end{gathered}$$

where "$_{|}j$" and "$|j$" indicate the first and second covariant differentiations of Cartan respectively and R_{ijkh} are the components of the third curvature tensor of Cartan.

Let N be endowed with the D-connection. This connection is defined by

(2.7) $$\begin{gathered}\Gamma=(w_B^A)=\begin{pmatrix}\omega_b^a & \omega_{n+\beta}^a\\ \omega^{n+\alpha}_b & \omega_{n+\beta}^{n+\alpha}\end{pmatrix},\qquad \omega_{n+\beta}^{n+\alpha}=\omega_\beta^\alpha,\quad \omega^{n+\alpha}_b=-\omega_{n+\alpha}^b.\\ \omega^{n+\alpha}_b=(A_j{}^i{}_k+R_h{}^i{}_{jk}l^h)\zeta_i^\alpha\zeta_b^j\zeta_c^k\omega^c-\sum_\gamma A_j{}^i{}_{k\,|\,h}l^h\zeta_i^\alpha\zeta_b^j\zeta_\gamma^k\omega_\gamma.\end{gathered}$$

As to the forms ω^a_b in (2.6) (or(2.7)), it is known [13] that these are the connection forms of Cartan and $\omega^a_b=-\omega^b_a$. In this case, N becomes a $(2n-1)$-dimensional Riemannian manifold with the metric whose components are δ_{AB} with respect to the frame (e_A) and the D-connection is metrical but not symmetric in general. Further the following hold good:

$$\langle e_A, e_B\rangle=\delta_{AB},\quad \langle\omega^A, \omega^B\rangle=\delta^{AB}\quad (\langle\ ,\rangle;\ \text{inner product}).$$

Thus, M is realizable as an n-dimensional distribution V on N which is defined by $\omega_\alpha=0$ $(\alpha=1, 2, \ldots, n-1)$.

§3. Involutive distributions. In the following, we assume that M is a space with the absolute parallelism of Cartan, that is

(3.1) $$R_h{}^i{}_{jk}l^h=0.$$

Then, it firstly follows from the condition (3.1) that the D-connection becomes Riemannian, and secondly that the distribution V is involutive, and thus the theorem in §1 holds good.

A system of differential equations

(3.2) $$\omega_\alpha=-\zeta^\alpha_i Dl^i=0 \quad \text{or} \quad Dl^i=dl^i+\Gamma^*{}_j{}^i{}_k l^j dx^k=0$$

is completely integrable, and a local base for V is given by (e_a) $(a=1, 2, \ldots, n)$, which is also a local base for M.

On the other hand, a system of differential equations

(3.3) $$\omega^a=\zeta^a_i dx^i=0 \quad \text{or} \quad dx^i=0$$

defines evidently an $(n-1)$-dimensional involutive distribution on N, which is the orthogonal complement of V. We denote it by $V^\perp$. A local base for the $V^\perp$ is given by $(e_{n+\alpha})$ $(\alpha=1, 2, \ldots, n-1)$, which is also a local base for the figuratrix on M.

Now, when we consider any element $(x, l^*)\in N$, there exist respective integral manifolds through (x, l^*) of (3.2) and (3.3) such that they are orthogonal to each other, and one represents a domain of M containing the point x, the other the figuratrix of M at x. In the sequel, we shall denote by V and $V^\perp$ such manifolds again for the sake of brevity.

A distribution E on N is said to be *parallel* if, for any vector field X on N and any vector field Y belonging to E, $\nabla_X Y$ belongs always to E, where ∇ means the covariant differentiation with respect to D-connection.

Now, suppose that the distribution V is parallel. Then, since $\nabla_{e_B} e_A=\Gamma_A{}^D{}_B e_D$, from the above definition we have $\nabla_{e_B} e_b=\Gamma_b{}^a{}_B e_a$ and hence

(3.4) $$\Gamma_b{}^{n+\alpha}{}_B=0.$$

On the other hand, since $\omega^{n+\alpha}_b=\Gamma_b{}^{n+\alpha}{}_B\,\omega^B$, if follows from (2.7), (3.1) and (3.4) that $A_j{}^i{}_k\zeta^\alpha_i\zeta^j_b\zeta^k_c=0$ and accordingly, by virtue of (2.2) and (2.3), $A_j{}^i{}_k=0$. In this case, the distribution $V^\perp$ is also parallel.

Hence we have

THEOREM 1. *If the distribution V (or $V^{\perp}$) is parallel, then M becomes a Riemannian manifold.*

Conversely if $A_j{}^i{}_k=0$, it follows that $\omega^{n}{}_b{}^{+\alpha}=\omega_{n+\alpha}^{b}=0$ and hence the distributions V and $V^{\perp}$ are both parallel. Consequently we have

COROLLARY 1.1. *Let M be an n-dimensional Riemannian manifold endowed with a parallel vector field l^i. Then, the sphere bundle N on M can be a $(2n-1)$-dimensional Riemannian manifold and becomes locally a product space of the submanifolds V and $V^{\perp}$.*

Note. Let L be the indicatrix bundle on M. Then, we have a natural isometry ϕ of L onto N defined by

$$\phi:\ (x, l)\rightarrow(x, l^*),\quad l_i^*=g_{ij}l^j.$$

By this isometry we can identify N with L. Corollary 1.1 is stated as above under the identification of N with L.

§4. Totally geodesic, umbilic and minimal submanifolds. First we shall find the Euler-Schouten tensor of the submanifold V of the Riemannian manifold N.

For every pair of e_a and e_b in the base for V, we have

$$\nabla_{e_B}e_a=\Gamma_a{}^D{}_b e_D=\Gamma_a{}^c{}_b e_c+\Gamma_a{}^{n+\alpha}{}_b e_{n+\alpha}. \tag{4.1}$$

If we denote by ∇' and $h_a{}^{n+\alpha}{}_b$ the covariant differentiation in V and components of the Euler-Schouten tensor of V, from (4.1) we have

$$\nabla'_{e_B}e_a=\Gamma_a{}^c{}_b e_c,\quad h_a{}^{n+\alpha}{}_b=\Gamma_a{}^{n+\alpha}{}_b. \tag{4.2}$$

By virtue of (2.7), (3.1), (3.2) and (4.2) we have

$$h_a{}^{n+\alpha}{}_b=A_j{}^i{}_k\zeta_i^\alpha\zeta_a^j\zeta_b^k, \tag{4.3}$$

which is the required result.

Then, it follows from (2.2), (2.3) and (4.3) that $h_a{}^{n+\alpha}{}_b=0$ if and only if $A_j{}^i{}_k=0$, that is, M becomes Riemannian. Since the components of the metric tensor of V are δ_{ab} with respect to the frame (e_a), V is totally umbilic in N if and olny if

$$h_a{}^{n+\alpha}{}_b=C^{n+\alpha}\delta_{ab}. \tag{4.4}$$

Contracting (4.4) by δ^{ab} and using (4.3), we have

$$C^{n+\alpha}=A^i\zeta_i^\alpha/n, \tag{4.5}$$

which is the mean curvature vector of V. If we substitute (4.5) in (4.4) and use (4.3) again, we have $A_j{}^i{}_k=A^ig_{jk}/n$, which implies $A_j{}^i{}_k=0$ and hence M becomes Riemannian. Further the condition $C^{n+\alpha}=0$ leads us to $A^i=0$ and accordingly M becomes Riemannian owing to the theorem of Deicke [2].

Thus we have

THEOREM 2. *If any of the following conditions holds good, then M becomes Riemannian*:

(1) *V is totally geodesic in N.*
(2) *V is totally umbilic in N.*
(3) *V is minimal in N.*

If $A_j{}^i{}_k=0$, the Euler-Schouten tensor vanishes. Consequently we have

COROLLARY 2.1. *If M is a Riemannian manifold with a parallel vector field l^i, then the submanifold V is always totally geodesic in the sphere bundle N.*

Next, we shall consider the submanifold $V^{\perp}$. For every pair of $e_{n+\alpha}$ and $e_{n+\beta}$ in the base for $V^{\perp}$, we have

$$\nabla_{e_{n+\beta}}\, e_{n+\alpha}=\Gamma_{n+\alpha}{}^{A}{}_{n+\beta}e_{A}=\Gamma_{n+\alpha}{}^{a}{}_{n+\beta}e_{a}+\Gamma_{n+\alpha}{}^{n+\gamma}{}_{n+\beta}e_{n+\gamma},$$

which leads us to

$$\nabla''_{e_{n+\beta}}\, e_{n+\alpha}=\Gamma_{n+\alpha}{}^{n+\gamma}{}_{n+\beta}e_{n+\gamma},\quad h_{n+\alpha}{}^{a}{}_{n+\beta}=\Gamma_{n+\alpha}{}^{a}{}_{n+\beta}, \tag{4.6}$$

where ∇'' denotes the covariant differentiation in $V^{\perp}$ and $h_{n+\alpha}{}^{a}{}_{n+\beta}$ are the components of the Euler-Schouten tensor of $V^{\perp}$. Then, it follows from (2.7), (3.1), (3.3) and (4.6) that

$$h_{n+\alpha}{}^{a}{}_{n+\beta}=A_j{}^i{}_{kl\mid o}\zeta^{a}_{i}\zeta^{j}_{\alpha}\zeta^{k}_{\beta}, \tag{4.7}$$

where the index o indicates the contraction by l^i. Immediately it is seen from (4.7) that $h_{n+\alpha}{}^{a}{}_{n+\beta}=0$ if and only if $A_j{}^i{}_{k\mid o}=0$. Since the components of the metric tensor of $V^{\perp}$ are $\delta_{n+\alpha\ n+\beta}$ with respect to the frame $(e_{n+\alpha})$, $V^{\perp}$ is totally umbilic in N if and only if

$$h_{n+\alpha}{}^{a}{}_{n+\beta}=C^{a}\delta_{n+\alpha\ n+\beta}, \tag{4.8}$$

from which it follows

$$C^{a}=A^{i}{}_{\mid o}\zeta^{a}_{i}/(n-1), \tag{4.9}$$

which is the mean curvature vector of $V^{\perp}$. From (4.7), (4.8) and (4.9) we have

$$A_j{}^i{}_{k\mid o}=A^{i}{}_{\mid o}h_{jk}/(n-1). \tag{4.10}$$

If (4.10) is summed with respect to i and k, it follows $(n-2)A_{j\mid o}=0$. Consequently, when $n>2$, we have $A_{j\mid o}=0$ and hence $A_j{}^i{}_{k\mid o}=0$. The condition $C^a=0$ implies $A^i{}_{\mid o}=0$, too. Thus we have

THEOREM 3. *For the submanifold $V^{\perp}$, the following hold good*:

(1) *$V^{\perp}$ is totally geodesic in N if and only if M is a Landsberg space.*
(2) *For $n>2$, $V^{\perp}$ is totally umbilic in N if and only if $V^{\perp}$ is totally geodesic in N.*
(3) *$V^{\perp}$ is minimal in N if and only if $A^i{}_{\mid o}=0$.*

Note. For $n=2$, since dim $V^{\perp}=1$, the relation (4.8) is always valid. Consequently the relation (4.10) will be satisfied identically. In fact, we have [11]

$$A_{ijk}=A_i h_{jk}, \quad A_i=Im_i,$$

where I is the main scalar and m_i is a unit vector orthogonal to l_i, and hence (4.10) is true.

§5. Osculating Riemannian spaces. We put

$$G^i(x,y)=(1/2)\gamma_j{}^i{}_k(x,y)y^j y^k \tag{5.1}$$

where $\gamma_j{}^i{}_k(x,y)$ are Christoffel symbols in M. If we further put $G^i_j(x,y)=\partial G^i(x,y)/\partial y^j$, we have [11]

$$G^i_j(x,y)=\Gamma^*{}_j{}^i{}_k y^k. \tag{5.2}$$

Since $G^i_j(x,y)$ are homogeneous of degree 1 in y^i, it follows from (3.2) and (5.2) that

$$\frac{\partial l^i}{\partial x^j}=-G^i_j(x,l)^{1)}, \tag{5.3}$$

which is, inconsequence of (3.1), completely integrable. Therefore, there exists a solution $l=l(x)$ such that $g_{ij}l^i l^j=1$ along the solution and $l^i=l^i_o$ when $x^i=x^i_o$ (l^i_o, x^i_o: any constants). In this case, the Finsler space M can be locally an osculating Riemannian space with the metric tensor $g_{ij}(x, l(x))$. Such a space will be denoted by M_r[2]. If we put $G_j{}^i{}_k(x,y)=\partial G_j{}^i(x,y)/\partial y^k$, being homogeneous of degree 0 in y^i, (3.1) is expressible in

$$\text{(3.1)}' \quad \frac{\partial}{\partial x^k}G^i_j(x,l)-\frac{\partial}{\partial x^j}G^i_k(x,l)-G_j{}^i{}_r(x,l)G^r_k(x,l)+G_k{}^i{}_r(x,l)G^r_j(x,l)=0.^{3)}$$

Now, if we denote by $\{{}_j{}^i{}_k\}$ and $\overset{r}{R}_j{}^i{}_{kl}$ the Christoffel symbols and the components of the curvature tensor on M_r, it follows from (5.2), (5.3) and (3.1)′ that

$$\{{}_j{}^i{}_k\}=\Gamma^*{}_j{}^i{}_k(x,l(x)), \quad \overset{r}{R}_j{}^i{}_{kl}=R_j{}^i{}_{kl}(x,l(x)). \tag{5.4}$$

And further for a proper tensor on M, for example, $T^i_j(x,y)=T^i_j(x,l)$, we have

$$\nabla_k T^i_j(x,l(x))=T^i_{j|k}(x,l(x)), \tag{5.5}$$

where ∇_k denotes the covariant differentiation in M_r.

First, let M_r be an Einstein space, that is

$$\overset{r}{R}_{jk}=\overset{r}{R}_j{}^i{}_{ki}=k(x)g_{jk}. \tag{5.6}$$

Then, if we contract (5.6) by l^j, we have $k(x)=0$ because of (3.1) and hence

1) 3) (5.3) and (3.1)′ are derived from another view point [12].

2) The extremal fields of M_r possess a group of translations whose paths are extremals [12].

$\overset{r}{R}_{jk}=0$.

Next, if M_r is of constant curvature, that is

$$\overset{r}{R}_j{}^i{}_{kl}=K(g_{jk}\delta^i_l-g_{jl}\delta^i_k),$$

it follows as before that $K=0$ or $l_k\delta^i_l-l_l\delta^i_k=0$. The second condition implies $l_i=0$.

Finally, we shall seek for a condition for M_r to be conformally flat. The components $\overset{r}{C}_j{}^i{}_{kl}$ of the conformal curvature tensor of Weyl are given by [5]

$$(5.7)\qquad \overset{r}{C}_j{}^i{}_{kl}=\overset{r}{R}_j{}^i{}_{kl}-\frac{1}{n-2}\overset{r}{R}_{jk}\delta^i_l-(\overset{r}{R}_{jl}\delta^i_k+g_{jk}\overset{r}{R}^i_l-g_{jl}\overset{r}{R}^i_k)$$
$$+\frac{\overset{r}{R}}{(n-1)\ (n-2)}(g_{jk}\delta^i_l-g_{jl}\delta^i_k),$$

where $\overset{r}{R}^i_l=g^{ij}\overset{r}{R}_{jl}$ and $\overset{r}{R}=\overset{r}{R}^i_i$. Suppose $\overset{r}{C}_j{}^i{}_{kl}=0$. Then, contracting (5.7) by l^il^k and using (3.1), we get

$$(5.8)\qquad \overset{r}{R}_{il}=\frac{1}{n-1}\overset{r}{R}h_{il},\qquad \overset{r}{R}^i_l=\frac{1}{n-1}\overset{r}{R}h^i_l,$$

where $h_{il}=g_{il}-l_il_l$ and $h^i_l=g^{ij}h_{jl}$. Substituting (5.8) in (5.7), we have

$$(5.9)\qquad \begin{aligned}\overset{r}{R}_j{}^i{}_{kl}&=\frac{\overset{r}{R}}{(n-1)(n-2)}(h_{jk}\delta^i_l-h_{jl}\delta^i_k-g_{jk}l^il_l+g_{jl}l^il_k),\\ \overset{r}{R}_{jikl}&=\frac{\overset{r}{R}}{(n-1)(n-2)}(h_{jk}g_{il}-h_{jl}g_{ik}-g_{jk}l_il_l+g_{jl}l_il_k).\end{aligned}$$

In consequence of (5.5), the Bianchi's identity

$$\nabla_h\overset{r}{R}_{jikl}+\nabla_l\overset{r}{R}_{jihk}+\nabla_k\overset{r}{R}_{jilh}=0$$

is rewritten in

$$(5.10)\qquad \overset{r}{R}_{jikl\,|\,h}+\overset{r}{R}_{jihk\,|\,l}+\overset{r}{R}_{jilh\,|\,k}=0$$

Contracting (5.10) by $g^{il}g^{jk}$ and using (5.8), we have

$$(5.11)\qquad (n-3)\overset{r}{R}_{|\,h}+2\overset{r}{R}_{|}l_{oh}=0,$$

which implies $\overset{r}{R}_{|\,o}=0$. When $n\geqq 4$, it follows from (5.11) that $\overset{r}{R}_{|\,h}=0$, that is, $\overset{r}{R}$ is constant.

In the case $n=3$, instead of $\overset{r}{C}_j{}^i{}_{kl}$ we deal with

$$\overset{r}{C}_{ijk}=\frac{1}{n-2}(\overset{r}{R}_{ij|k}-\overset{r}{R}_{ik|j})-\frac{1}{2(n-1)(n-2)}(\overset{r}{R}_{ij}\overset{r}{R}_{|k}-\overset{r}{R}_{ik}\overset{r}{R}_{|j}). \tag{5.12}$$

Suppose $\overset{r}{C}_{ijk}=0$. Then, since the tensor $\overset{r}{C}_j{}^i{}_{kl}$ vanishes automatically, the conditions (5.8) and (5.9) still hold. Substituting (5.8) in (5.12), we have

$$\{2(n-1)-\overset{r}{R}\}(h_{ij}\overset{r}{R}_{|k}-h_{ik}\overset{r}{R}_{|j})=0,$$

from which it follows that $\overset{r}{R}=2(n-1)$ or

$$h_{ij}\overset{r}{R}_{|k}-h_{ij}\overset{r}{R}_{|j}=0. \tag{5.13}$$

Contracting (5.13) by g^{ij}, we have

$$(n-1)\overset{r}{R}_{|k}+\overset{r}{R}_{|o}l_k=0,$$

which leads us to $\overset{r}{R}_{|k}=0$. Conversely, the condition (5.9) implies $\overset{r}{C}_j{}^i{}_{kl}=0$, and if (5.9) holds and $\overset{r}{R}$ is constant, it follows $\overset{r}{C}_{ijk}=0$. And in any case, we have $\nabla_h\overset{r}{R}_j{}^i{}_{kl}=0$. Thus, making a summary of the results obtained, we have

THEOREM 4. *For the space M_r, the following hold good:*
(1) *If M_r is a Einstein space, the Ricci tensor vanishes.*
(2) *If M_r is of constant curvature, M_r is locally flat.*
(3) *When $n\geqq 4$, M_r is conformally flat if and only if the curvature tensor is expressed in* (5.9). *In this case, the scalar curvature is constant. When $n=3$, M_r is conformally flat if and only if the curvature tensor is expressed in* (5.9) *and the scalar curvature is constant. In any case, if M_r is conformally flat, M_r is locally symmetric.*

§6. Spaces of constant curvature $K=0$. If we put $G_j{}^i{}_{hk}=\partial G_j{}^i{}_h/\partial y^k$, the curvature tensor of Berwald is given by [1]

$$H_j{}^i{}_{hk}=\frac{\partial G_j{}^i{}_h}{\partial x^k}-\frac{\partial G_j{}^i{}_k}{\partial x^h}+G_j{}^r{}_hG_r{}^i{}_k-G_j{}^r{}_kG_r{}^i{}_h+G_r{}^i{}_{jk}G_h{}^r-G_r{}^i{}_{jh}G_k{}^r,$$

which is expressible in [11]

$$H_j{}^i{}_{hk}=R_j{}^i{}_{hk}+g^{ir}(y^s\partial R_{srhk}/\partial y^j-2A_j{}^m{}_rR_{smhk}l^s)=\partial(R_s{}^i{}_{hk}y^s)/\partial y^j. \tag{6.1}$$

On the other hand, the curvature tensor R_{ijkh} is expressible in [10]

$$R_{ijkh}=(H_{ijkh}-H_{jikh})/2-(A_{irk|o}A_j{}^r{}_{h|o}-A_{jrh|o}A_i{}^r{}_{k|o}). \tag{6.2}$$

As is well known [1], [11], M is of constant curvature K if and only if

(6.3) $$H_{hijk}=K(g_{hj}g_{ik}-g_{hk}g_{ij}).$$

Now, let M be of constant curvature $K=0$. Then it follows from (6.2) and (6.3) that the tensor R_{ijhk} is written in

(6.4) $$R_{ijhk}=A_{irh|o}A_j{}^r{}_{k|o}-A_{irk|o}A_j{}^r{}_{h|o}.$$

Consequently, by virtue of (6.1) and (6.4) we see that $H_{hijk}=0$ is equivalent to $R_s{}^i{}_{jk}y^s=0$, that is, M is of constant curvature $K=0$ if and only if M is a space with the absolute parallelism of line-elements in the sense of Cartan. Therefore we can apply the results in § 4 to the space M. Immediately from (6.4) and Theorem 3 we have

THEOREM 5. *Let M be of constant curvature $K=0$. Then, the following hold good:*
(1) *If $V^\perp$ is totally geodesic in N, the curvature tensor R_{ijhk} vanishes.*
(2) *For $n>2$, if $V^\perp$ is totally umbilic in N, the curvature tensor R_{ijhk} vanishes.*

Next, we shall consider a C-reducible Finsler space [9]. Such a space is characterized by

$$A_{ijk}=(h_{ij}A_k+h_{kj}A_i+h_{ki}A_j)/(n+1),$$

provided that $h_{ij}=g_{ij}-l_il_j$ and the dimension of the space is more than 2. And it is known [9], [10] that the following conditions are equivalent:

(6.5) $$(1)\ A_{ijk|h}=0 \quad (2)\ A_{ijk|o}=0 \quad (3)\ A_{i|o}=0,$$

and that

(6.6) $$-R_{ijkh}=-(H_{ijkh}-H_{ijhk})/2+h_{ik}H_{jh}+h_{jh}H_{ik}-h_{ih}H_{jk}-h_{jk}H_{ih},$$

$$\text{where}\quad H_{ik}=(A_{i|o}A_{k|o}+(1/2)\mu h_{ik})/(n+1)^2 \text{ and } \mu=A_{r|o}A^r{}_{|o}.$$

Then, if M is of constant curvature $K=0$ and $V^\perp$ is minimal in N, it follows from (6.2), (6.5), (6.6) and Theorem 3 that $R_{ijhk}=0$ and $A_{ijh|k}=0$, that is, M is locally Minkowskian. Conversely, the conditions $R_{ijhk}=0$ and $A_{ijk|h}=0$ lead us to $H_{ijhk}=0$ and $A_{i|o}=0$.

Hence we have

THEOREM 6. *Let M be a C-reducible Finsler space. Then, M is locally Minkowskian if and only if M is of constant curvature $K=0$ and V is minimal in N.*

If $K=0$, the expression (6.6) is reducible to

(6.7) $$-R_{ijhk}=h_{ik}H_{jh}+h_{jh}H_{ik}-h_{ih}H_{jk}-h_{jk}H_{ih}.$$

Contracting (6.7) by g^{ik}, we have

(6.8) $$-(n+1)^2R_{ik}=(n-1)\mu h_{ik}+(n-3)A_{i|o}A_{k|o}.$$

Suppose $R_{ik}=0$. Then the contraction of (6.8) by g^{ik} yields $\mu=0$. For $n\geqq 4$, (6.8) leads us to $A_{i|o}=0$. For $n=3$, if the metric is positive definite, the condition $\mu=0$ implies $A_{i|o}=0$.

Suppose $R=0$. Then, if the metric is positive definite, it follows as before that $A_{i|o}=0$. Consequently we have

THEOREM 7. *Let M be a C-reducible Finsler space of constant curvature $K=0$. Then, M becomes locally Minkowskian if any of the following conditions holds good:*

(1) *M is a 3-dimensional space such that the metric is positive definite and the Ricci tensor R_{ij} vanishes.*

(2) *The dimension of M is more than 3 and the Ricci tensor R_{ij} vanishes.*

(3) *The metric is positive definite and the scalar R vanishes.*

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