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NORMAL CONDITIONS ON A HYPERSURFACE OF A QUATERNIONIC KAEHLERIAN MANIFOLD

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§ 0. Introduction.

Recently, K. Yano, U-H. Ki and one of the present authors proved the following [3]

THEOREM A. *On a hypersurface of an almost quaternion manifold, the condition* $[F,F]+du\otimes U=0$ *and* $[G,G]+dv\otimes V=0$ *and the condition* $[F,G]+du\otimes V+dv\otimes U=0$ *are equivalent, where* (F,U,u), (G,V,v) *and* (H,W,w) *are the almost contact three structures induced on the hypersurface.*

The purpose of the present paper is to prove Theorem A in the case of a hypersurface of a quaternionic Kaehlerian manifold.

§ 1. Preliminaries.

Let $'M^{4n}$ be an almost quaternionic manifold, that is, a $4n$-dimensional differentiable manifold which admits a set of three tensor fields $'F$, $'G$, $'H$ of type (1, 1) satisfying

$$
\begin{gathered}
'F^2=-I, \quad 'G^2=-I, \quad 'H^2=-I, \\
'F='G'H=-'H'G, \ 'G='H'F=-'F'H, \ 'H='F'G=-'G'F,
\end{gathered} \tag{1.1}
$$

I denoting the identity tensor.

In a previous paper [3], we proved that there exists a Hermitian metric $'g$ for the almost quaternionic structure $'F$, $'G$, $'H$, that is, a Riemannian metric $'g$ satisfying

$$
\begin{aligned}
&'g('FX,\ 'FY)='g(X,Y),\\
&'g('GX,\ 'GY)='g(X,Y),\\
&'g('HX,\ 'HY)='g(X,Y)
\end{aligned} \tag{1.2}
$$

for arbitrary vector field X and Y of $'M^{4n}$. In this case $'M^{4n}$ is called an *almost quaternionic metric manifold.*

If an almost quaternionic metric manifold $'M^{4n}$ satisfies the condition

$$
\begin{aligned}
'\nabla_X{'F} &= \qquad\qquad 'r(X)'G-'q(X)'H,\\
'\nabla_X{'G} &= -'r(X)'F \qquad\qquad +'p(X)'H,\\
'\nabla_X{'H} &= \ 'q(X)'F-'p(X)'G,
\end{aligned} \tag{1.3}
$$

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where $'\nabla$ is the operator of covariant differentiation with respect to $'g$ and $'p$, $'q$, $'r$ are certain 1-forms, X being an arbitrary vector field of $'M^{4n}$, then $'M^{4n}$ is called a *quaternionic Kaehlerian manifold* [1].

Suppose that a $(4n-1)$-dimensional orientable differentiable manifold M^{4n-1} is immersed differentiably in an almost quaternionic metric manifold $'M^{4n}$ by the immersion i: $M^{4n-1} \longrightarrow 'M^{4n}$ and denote by B the differential of i. We denote by C the unit normal to $i(M^{4n-1})$ with respect to the Hermitian metric $'g$ introduced above.

Then the transform of a vector field tangent to $i(M^{4n-1})$ and that of the unit normal vector field by $'F$, $'G$ and $'H$ can be expressed respectively as

$$
\begin{aligned}
&'FBX=BFX+u(X)C, \qquad &'FC=-BU,\\
&'GBX=BGX+v(X)C, \qquad &'GC=-BV,\\
&'HBX=BHX+w(X)C, \qquad &'HC=-BW,
\end{aligned} \tag{1.4}
$$

where F, G and H are tensor fields of type $(1,1)$, U, V, W vector fields, u, v, w 1-forms and X an arbitrary vector field of M^{4n-1}.

In the previous paper [3], we proved that (F, U, u), (G, V, v) and (H, W, w) are three almost contact metric structures, that is, they satisfy the following equations.

$$
\begin{aligned}
&F^2=-I+u\otimes U, \quad G^2=-I+v\otimes V, \quad H^2=-I+w\otimes W,\\
&GH=F+w\otimes V, \quad HF=G+u\otimes W, \quad FG=H+v\otimes U,\\
&HG=-F+v\otimes W, \quad FH=-G+w\otimes U, \quad GF=-H+u\otimes V,
\end{aligned} \tag{1.5}
$$

$$
\begin{aligned}
&u\circ F=0, \qquad u\circ G=w, \qquad u\circ H=-v,\\
&v\circ F=-w, \quad v\circ G=0, \qquad v\circ H=u,\\
&w\circ F=v, \qquad w\circ G=-u, \quad w\circ H=0,
\end{aligned} \tag{1.6}
$$

$$
\begin{aligned}
&FU=0, \qquad FV=W, \qquad FW=-V,\\
&GU=-W, \quad GV=0, \qquad GW=U,\\
&HU=V, \qquad HV=-U, \quad HW=0,
\end{aligned} \tag{1.7}
$$

$$
\begin{aligned}
&u(U)=1, \quad u(V)=0, \quad u(W)=0,\\
&v(U)=0, \quad v(V)=1, \quad v(W)=0,\\
&w(U)=0, \quad w(V)=0, \quad w(V)=1,
\end{aligned} \tag{1.8}
$$

$$
\begin{aligned}
&g(FX, FY)=g(X, Y)-u(X)u(Y),\\
&g(GX, GY)=g(X, Y)-v(X)v(Y),\\
&g(HX, HY)=g(X, Y)-w(X)w(Y),
\end{aligned} \tag{1.9}
$$

where g is the Riemannian metric on M^{4n-1} induced from that of $'M^{4n}$, that is,

$'g(BX, BY)=g(X, Y)$.

Let $\{'U; y^\lambda\}$ and $\{U; x^h\}$ be the coordinate neighborhoods of $'M^{4n}$ and M^{4n-1} respectively, where, here and in the sequel, the indices $\lambda, \mu, \nu, \tau, \ldots\ldots$ run over the range $\{1, 2, \ldots, 4n\}$ and the indices $h, i, j, k, \ldots\ldots$ the range $\{1, 2, \ldots, 4n-1\}$.

For the hypersurface $i(M^{4n-1})$, the equations of Gauss and Weingarten are respectively

$$\nabla_k B_j{}^\lambda = h_{kj} C^\lambda, \quad \nabla_k C^\lambda = -h_k{}^i B_i{}^\lambda, \tag{1.10}$$

where $B_j{}^\lambda = \partial_j y^\lambda$ $(\partial_j = \partial/\partial x^j)$, h_{kj} is the second fundamental tensor of $i(M^{4n-1})$ and $h_k{}^i = h_{kj} g^{ji}$.

Differentiating (1.4) covariantly along the hypersurface $i(M^{4n-1})$ of a quaternionic Kaehlerian manifold and taking account of (1.3) and (1.10), we obtain

$$\begin{aligned} \nabla_k F_j{}^i &= r_k G_j{}^i - q_k H_j{}^i - h_{kj} U^i + h_k{}^i u_j, \\ \nabla_k G_j{}^i &= p_k H_j{}^i - r_k F_j{}^i - h_{kj} V^i + h_k{}^i v_j, \\ \nabla_k H_j{}^i &= q_k F_j{}^i - p_k G_j{}^i - h_{kj} W^i + h_k{}^i w_j, \end{aligned} \tag{1.11}$$

$$\begin{aligned} \nabla_k u_j &= r_k v_j - q_k w_j - h_{kt} F_j{}^t, \\ \nabla_k v_j &= p_k w_j - r_k u_j - h_{kt} G_j{}^t, \\ \nabla_k w_j &= q_k u_j - p_k v_j - h_{kt} H_j{}^t, \end{aligned} \tag{1.12}$$

where $F_j{}^i$, $G_j{}^i$ and $H_j{}^i$ are respectively components of F, G and H, u_j, v_j and w_j those of u, v and w and

$$p_k = 'p_\lambda B_k{}^\lambda, \quad q_k = 'q_\lambda B_k{}^\lambda, \quad r_k = 'r_\lambda B_k{}^\lambda.$$

§2. Normal conditions on a hypersurface of a quaternionic Kaehlerian manifold.

In this section, we consider the normal conditions on a hypersurface M^{4n-1} of a quaternionic Kaehlerian manifold $'M^{4n}$ $(n \geq 2)$.

The almost contact structure is said to be normal if the tensor

$$[F, F] + du \otimes U$$

vanishes, where $[F, F]$ is the Nijenhuis tensor formed with F.

We compute components of this tensor.

$$\begin{aligned} &[F, F]_{ji}{}^h + (\nabla_j u_i - \nabla_i u_j) U^h \\ &= (r_t F_j{}^t - q_j) G_i{}^h - (r_t F_i{}^t - q_i) G_j{}^h \\ &\quad - (q_t F_j{}^t + r_j) H_i{}^h + (q_t F_i{}^t + r_i) H_j{}^h \\ &\quad + (F_j{}^t h_t{}^h - h_j{}^t F_t{}^h) u_i - (F_i{}^t h_t{}^h - h_i{}^t F_t{}^h) u_j. \end{aligned} \tag{2.1}$$

Similarly, computing components of the tensors

$$[G,G]+dv\otimes V \text{ and } [H,H]+dw\otimes W$$

respectively, we find

$$\begin{aligned}&[G,G]_{ji}{}^h+(\nabla_j v_i-\nabla_i v_j)V^h\\ (2.2)\quad &=(p_tG_j{}^t-r_j)H_i{}^h-(p_tG_i{}^t-r_i)H_j{}^h\\ &\quad -(r_tG_j{}^t+p_j)F_i{}^h+(r_tG_i{}^t+p_i)F_j{}^h\\ &\quad +(G_j{}^th_t{}^h-h_j{}^tG_t{}^h)v_i-(G_i{}^th_t{}^h-h_i{}^tG_t{}^h)v_j,\end{aligned}$$

$$\begin{aligned}&[H,H]_{ji}{}^h+(\nabla_j w_i-\nabla_i w_j)W^h\\ (2.3)\quad &=(q_tH_j{}^t-p_j)F_i{}^h-(q_tH_i{}^t-p_i)F_j{}^h\\ &\quad -(p_tH_j{}^t+q_j)G_i{}^h+(p_tH_i{}^t+q_i)G_j{}^h\\ &\quad +(H_j{}^th_t{}^h-h_j{}^tH_t{}^h)w_i-(H_i{}^th_t{}^h-h_i{}^tH_t{}^h)w_j.\end{aligned}$$

We also compute components of the tensor

$$[F,G]+du\otimes V+dv\otimes U.$$

$$\begin{aligned}&[F,G]_{ji}{}^h+(\nabla_j u_i-\nabla_i u_j)V^h+(\nabla_j v_i-\nabla_i v_j)U^h\\ (2.4)\quad &=(p_tF_j{}^t-q_tG_j{}^t)H_i{}^h-(p_tF_i{}^t-q_tG_i{}^t)H_j{}^h\\ &\quad +(r_tG_j{}^t+p_j)G_i{}^h-(r_tG_i{}^t+p_i)G_j{}^h\\ &\quad -(r_tF_j{}^t-q_j)F_i{}^h+(r_tF_i{}^t-q_i)F_j{}^h\\ &\quad +(F_j{}^th_t{}^h-h_j{}^tF_t{}^h)v_i-(F_i{}^th_t{}^h-h_i{}^tF_t{}^h)v_j\\ &\quad +(G_j{}^th_t{}^h-h_j{}^tG_t{}^h)u_i-(G_i{}^th_t{}^h-h_i{}^tG_t{}^h)u_j.\end{aligned}$$

Suppose that the almost contact structures (F,U,u) and (G,V,v) are both normal. In this case, contracting with respect to j and h in (2.1), we find

$$(2.5)\qquad v_ir_tU^t-w_iq_tU^t-F_i{}^th_t{}^su_s=0.$$

Transvecting (2.5) with V^i and W^i respectively, we find

$$(2.6)\qquad r_tU^t=h_t{}^sW^tu_s=h_{ts}U^tW^s,$$

$$(2.7)\qquad q_tU^t=h_t{}^sV^tu_s=h_{ts}U^tV^s.$$

Transvecting (2.5) with $F_k{}^i$, we find

$$(2.8)\qquad h_k{}^su_s-h_{ts}U^tU^su_k-w_kr_tU^t-v_kq_tU^t=0.$$

Transvecting (2.8) with V^k, we find

$$(2.9)\qquad h_{ts}U^tV^s=q_tU^t.$$

Transvecting (2.1) with $G_h{}^k$ and contracting with respect to i and k, we find

$$(2.10)\qquad (-4n+4)(r_tF_j{}^t-q_j)-(r_tW^t-q_tV^t)v_j+q_tU^tu_j+(q_tW^t+r_tV^t)w_j +F_j{}^th_t{}^sw_s-h_j{}^tv_t=0.$$

Transvecting (2.10) with U^j, we find

$$(2.11)\qquad (4n-3)q_tU^t=h_j{}^tU^jv_t=h_{ts}U^tV^s.$$

Comparing (2.11) with (2.7), we have

$$(2.12)\qquad q_tU^t=0,\qquad h_{ts}U^tV^s=0.$$

Transvecting (2.10) with W^j, we find

$$(2.13)\qquad (4n-3)(r_tU^t+q_tW^t)-2h_{ts}V^tW^s=0.$$

On the other hand, contracting with respect to j and h in (2.2), we find

$$(2.14)\qquad w_ip_tV^t-u_ir_tV^t-G_i{}^th_t{}^sv_s=0.$$

Transvecting (2.14) with W^i and taking account of (2.12), we have

$$(2.15)\qquad p_tV^t=0.$$

Transvecting (2.14) with U^i, we find ·

$$(2.16)\qquad r_tV^t=h_t{}^sW^tv_s=h_{ts}V^tW^s.$$

Transvecting (2.2) with $H_h{}^k$ and contracting with respect to i and k, we find

$$(2.17)\qquad (-4n+4)(p_tG_j{}^t-r_j)+(r_tW^t-p_tU^t)w_j+r_tV^tv_j+(r_tU^t+p_tW^t)u_j +G_j{}^th_t{}^su_s-h_j{}^tw_t=0.$$

Transvecting (2.17) with V^j, we find

$$(2.18)\qquad (4n-3)r_tV^t=h_j{}^tV^jw_t=h_{ts}V^tW^s.$$

Comparing (2.18) with (2.16), we have

$$(2.19)\qquad r_tV^t=0,\qquad h_{ts}V^tW^s=0$$

Transvecting (2.10) with W^j and taking account of (2.19), we have

$$(2.20)\qquad q_tW^t=0.$$

Transvecting (2.17) with U^j, we find

$$(2.21)\qquad r_tU^t+p_tW^t-2h_t{}^sW^tu_s=0.$$

Transvecting (2.10) with V^j, we find

(2.22) $$(-4n+3)(r_tW^t-q_tV^t)+h_{ts}W^tW^s-h_{ts}V^tV^s=0.$$

Transvecting (2.17) with W^j, we find

(2.23) $$(-4n+3)(p_tU^t-r_tW^t)+h_{st}U^tU^s-h_{ts}W^tW^s=0.$$

Making (2.22)+(2.23), we find

(2.24) $$(-4n+3)(p_tU^t-q_tV^t)+h_{ts}U^tU^s-h_{ts}V^tV^s=0.$$

Substituting (2.19) and (2.20) into (2.13), we have

(2.25) $$r_tU^t=0.$$

Substituting (2.12) and (2.25) into (2.5), we find

(2.26) $$F_i^th_t^su_s=0.$$

Transvecting (2.26) with F_k^i, we have

(2.27) $$h_k^tu_t=\lambda u_k, \qquad \lambda=h_{ts}U^tU^s.$$

Substituting (2.15) and (2.19) into (2.14), we find

(2.28) $$G_i^th_t^sv_s=0.$$

Transvecting (2.28) with G_k^i, we have

(2.29) $$h_k^tv_t=\mu v_k, \qquad \mu=h_{ts}V^tV^s.$$

Taking account of (2.25) and (2.6), we have from (2.21)

(2.30) $$p_tW^t=0, \qquad h_{st}U^tW^s=0.$$

Transvecting (2.1) with V^i, we find

(2.31) $$(-r_tW^t+q_tV^t)G_j^h+(q_tF_j^t+v_j)U^h+(-W^th_t^h+\mu W^h)u_j=0.$$

Transvecting (2.31) with U^j and taking account of (2.29), we find

(2.32) $$(r_tW^t-q_tV^t)W^h-(W^th_t^h-\mu W^h)=0.$$

Transvecting (2.32) with w_h, we find

(2.33) $$r_tW^t-q_tV^t=h_{ts}W^tW^s-\mu.$$

Comparing (2.33) with (2.22), we have

(2.34) $$r_tW^t=q_tV^t, \qquad \mu=\nu, \qquad \nu=h_{ts}W^tW^s.$$

Substituting (2.34) into (2.32), we find

(2.35) $$h_t^kw_k=\mu w_t.$$

Transvecting (2.2) with W^i, we find

$$(2.36)\qquad (-p_tU^t+r_tW^t)H_j{}^h+(r_tG_j{}^t+p_j)V^h-(U^th_t{}^h-U^h)v_j=0.$$

Transvecting (2.2) with $W^iV^ju_h$ and taking account of (2.24) and (2.34), we have

$$p_tU^t=r_tW^t,\qquad \lambda=\mu.$$

Gathering above results, we have

$$(2.37)\qquad p_tU^t=q_tV^t=r_tW^t,\qquad p_tV^t=p_tW^t=q_tU^t=q_tW^t=r_tU^t=r_tV^t=0,$$

$$(2.38)\qquad h_{ts}U^tU^s=h_{ts}V^tV^s=h_{ts}W^tW^s.$$

Therefore we have from (2.31) and (2.36),

$$(2.39)\qquad q_tF_j{}^t=-r_j,\qquad r_tG_j{}^t=-p_j$$

by virtue of (2.27), (2.35), (2.37) and (2.38).

Transvecting the first equation of (2.39) with $F_k{}^j$, $G_k{}^j$ and $H_k{}^j$ respectively and taking account of (2.37), we have

$$(2.40)\qquad q_k=r_tF_k{}^t,\quad q_tH_k{}^t=-r_tG_k{}^t,\quad q_tG_k{}^t=r_tH_k{}^t.$$

Similarly from the second equation of (2.39) we have

$$(2.41)\qquad r_tH_k{}^t=p_tF_k{}^t,\quad r_k=p_tG_k{}^t,\quad r_tF_k{}^t=-p_tH_k{}^t.$$

Combining (2.39), (2.40) and (2.41), we obtain

$$(2.42)\qquad p_tF_k{}^t=q_tG_k{}^t=r_tH_k{}^t,$$

$$(2.43)\qquad p_k=q_tH_k{}^t=-r_tG_k{}^t,\qquad q_k=r_tF_k{}^t=-p_tH_k{}^t,\qquad r_k=p_tG_k{}^t=-q_tF_k{}^t.$$

On the other hand, transvecting (2.1) with U^i and taking account of (2.43) and (2.27), we easily see that

$$(2.44)\qquad F_j{}^th_t{}^h-h_j{}^tF_t{}^h=0.$$

Similarly, transvecting (2.2) with V^i and taking account of (2.43) and (2.29), we also see that

$$(2.45)\qquad G_j{}^th_t{}^h-h_j{}^tG_t{}^h=0.$$

Transvecting (2.44) with $G_k{}^j$ and taking account of (2.45), (2.27) and (2.29), we easily see that

$$(2.46)\qquad H_k{}^th_t{}^h-h_k{}^tH_t{}^h=0.$$

Substituting (2.43) and (2.46) into (2.3), we conclude that the almost contact structure (H, W, w) also is normal.

Thus we have the following

THEOREM 2.1. *Let* (F,U,u), (G,V,v) *and* (H,W,w) *are the almost contact three structures induced on a hypersurface of a quaternionic Kaehlerian manifold* $'M^{4n}$ $(n\geqq 2)$. *If two of the structures are normal, then the other also is normal.*

§ 3. Normal conditions on a hypersurface of a quaternionic Kaehlerian manifold. (continued)

In this section we consider the inverse case of § 2. We also assume that the dimension of a quaternionic Kaehlerian manifold is $4n$ and $n\geqq 2$.

If the almost contact structures (F,U,u) and (G,V,v) are both normal, then the equations (2.42), (2.43), (2.44) and (2.45) are satisfied. Substituting these equations into (2.4), we see that

$$[F,G]_{ji}{}^h+(\nabla_j u_i-\nabla_i u_j)V^h+(\nabla_j v_i-\nabla_i v_j)U^h=0. \tag{3.1}$$

Conversely, suppose that two almost contact structures (F,U,u) and (G,V,v) satisfy (3.1). In this case, contracting with respect to j and h in (3.1) and taking account of (2.4), we find

$$(p_tU^t-q_tV^t)w_i+r_tV^tv_i-r_tU^tu_i-F_i{}^th_t{}^sv_s-G_i{}^th_t{}^su_s=0. \tag{3.2}$$

Transvecting (3.2) with U^i and V^i respectively, we find

$$r_tU^t=h_t{}^su_sW^t=h_{ts}U^tW^s,\qquad r_tV^t=h_t{}^sv_sW^t=h_{ts}V^tW^s. \tag{3.3}$$

Transvecting (3.2) with W^i, we find

$$p_tU^t-q_tV^t+h_{ts}V^tV^s-h_{ts}U^tU^s=0. \tag{3.4}$$

Transvecting (3.1) with U^i and taking account of (2.4), we find

$$\begin{aligned}&(p_tF_j{}^t-q_tG_j{}^t)V^h-(r_tG_j{}^t+p_j)W^h-q_tW^tH_j{}^h+(r_tW^t-p_tU^t)G_j{}^h\\&-q_tU^tF_j{}^h+h_s{}^tU^sF_t{}^hv_j+(W^th_t{}^h+h_s{}^tU^sG_t{}^h)u_j+G_j{}^th_t{}^h-h_j{}^tG_t{}^h=0.\end{aligned} \tag{3.5}$$

Transvecting (3.5) with v_h, we find

$$p_tF_j{}^t-q_tG_j{}^t-q_tW^tu_j+q_tU^tw_j-h_s{}^tU^sw_tv_j+G_j{}^th_t{}^sv_s+W^th_t{}^sv_su_j=0. \tag{3.6}$$

Transvecting (3.6) with V^j, we find

$$p_tW^t=r_tU^t=h_{ts}U^tW^s \tag{3.7}$$

by virtue of (3.3).

Transvecting (3.6) with W^j, we find

$$p_tV^t=h_{ts}U^tV^s. \tag{3.8}$$

Transvecting (3.1) with $H_h{}^k$, taking account of (2.4) and contracting with respect to i and k, we find

$$(-4n+4)(p_tF_j{}^t-q_tG_j{}^t)+(p_tV^t+q_tU^t)w_j-p_tW^tv_j-q_tW^tu_j$$
$$+F_j{}^th_t{}^su_s-G_j{}^th_t{}^sv_s=0. \tag{3.9}$$

Transvecting (3.9) with V^j, we find

$$(4n-3)p_tW^t=h_s{}^tu_tW^s=h_{ts}U^tW^s. \tag{3.10}$$

Comparing (3.10) with (3.7), we have

$$p_tW^t=0,\quad r_tU^t=0,\quad h_{ts}U^tW^s=0. \tag{3.11}$$

Transvecting (3.9) with W^j, we find

$$(4n-3)(p_tV^t+q_tU^t)-2h_t{}^sV^tu_s=0. \tag{3.12}$$

Transvecting (3.9) with U^j, we find

$$(4n-3)q_tW^t-h_{ts}V^tW^s=0. \tag{3.13}$$

Transvecting (3.1) with V^i and taking account of (2.4), we find

$$-(p_tF_j{}^t-q_tG_j{}^t)U^h-(r_tF_j{}^t-q_j)W^h-q_tV^tG_j{}^h+(r_tW^t-q_tV^t)F_j{}^h$$
$$-(W^th_t{}^h-h_s{}^tV^sF_t{}^h)v_j+h_s{}^tV^sG_t{}^hu_j+F_j{}^th_t{}^h-h_j{}^tF_t{}^h=0. \tag{3.14}$$

Transvecting (3.14) with U^j, we find

$$-q_tW^tU^h+(p_tV^t+q_tU^t)W^h-h_s{}^tU^sF_t{}^h+h_s{}^tV^sG_t{}^h=0. \tag{3.15}$$

Transvecting (3.15) with u_h, we find

$$q_tW^t-h_{ts}V^tW^s=0. \tag{3.16}$$

Comparing (3.16) with (3.13) and taking account of (3.3), we have

$$q_tW^t=0,\quad r_tV^t=0,\quad h_{ts}V^tW^s=0. \tag{3.17}$$

Transvecting (3.15) with w_h, we find

$$p_tV^t+q_tU^t-2h_{ts}U^tV^s=0. \tag{3.18}$$

Comparing (3.12) with (3.18) and taking account of (3.8), we have

$$p_tV^t=0,\quad q_tU^t=0,\quad h_{ts}U^tV^s=0. \tag{3.19}$$

Transvecting (3.14) with v_h, we find

$$(-r_tW^t-q_tV^t)w_j+F_j{}^th_t{}^hv_h+h_j{}^tw_t=0. \tag{3.20}$$

Transvecting (3.20) with W^j, we find

$$q_tV^t-r_tW^t-h_{ts}V^tV^s+h_{ts}W^tW^s=0. \tag{3.21}$$

Transvecting (3.1) with $G_h{}^k$, taking account of (2.4) and contracting with respect to i and k, we find

$$(3.22)\qquad (-4n+4)(r_tG_j{}^t+p_j)-(p_tU^t-r_tW^t)u_j+G_j{}^th_t{}^sw_s+h_j{}^tu_t=0.$$

Transvecting (3.22) with U^j, we find

$$(3.23)\qquad (-4n+3)(p_tU^t-r_tW^t)+h_{ts}U^tU^s-h_{ts}W^tW^s=0.$$

Transvecting (3.1) with $F_h{}^k$, taking account of (2.4) and contracting with respect to i and k, we find

$$(3.24)\qquad (-4n+4)(r_tF_j{}^t-q_j)+(q_tV^t-r_tW^t)v_j+F_j{}^th_t{}^sw_s-h_j{}^tv_t=0.$$

Transvecting (3.24) with V^j, we find

$$(3.25)\qquad (-4n+3)(r_tW^t-q_tV^t)+h_{ts}W^tW^s-h_{ts}V^tV^s=0.$$

Making (3.23)+(3.25), we find

$$(3.26)\qquad (-4n+3)(p_tU^t-q_tV^t)+h_{ts}U^tU^s-h_{ts}V^tV^s=0.$$

Comparing (3.26) with (3.4), we find

$$(3.27)\qquad p_tU^t=q_tV^t,\qquad h_{ts}U^tU^s=h_{ts}V^tV^s.$$

Substituting (3.27) into (3.23) we find

$$(3.28)\qquad (-4n+3)(q_tV^t-r_tW^t)+h_{ts}V^tV^s-h_{ts}W^tW^s=0.$$

Comparing (3.21) and (3.28), we find

$$(3.29)\qquad q_tV^t=r_tW^t,\qquad h_{ts}V^tV^s=h_{ts}W^tW^s.$$

Transvecting (3.20) with $F_k{}^j$, we find

$$(3.30)\qquad -h_k{}^tv_t+F_k{}^jh_j{}^tw_t=0$$

by virtue of (3.29).
Substituting (3.30) and (3.29) into (3.24), we have

$$(3.31)\qquad q_j=r_tF_j{}^t.$$

Substituting (3.19) into (3.15). we find

$$(3.32)\qquad F_j{}^th_t{}^su_s-G_j{}^th_t{}^sv_s=0.$$

Substituting (3.11), (3.17), (3.19) and (3.32) into (3.9), we find

$$(3.33)\qquad p_tF_j{}^t-q_tG_j{}^t=0.$$

Transvecting (3.33) with $F_k{}^j$, we find

(3. 34) $$p_j=q_tH_j^{\ t}.$$

Transvecting (3. 31) with $F_k^{\ j}$, we find

(3. 35) $$r_j=-q_tF_j^{\ t}.$$

Transvecting (3. 34) with $H_k^{\ j}$, we find

(3. 36) $$q_j=-p_tH_j^{\ t}.$$

Transvecting (3. 33) with $H_k^{\ j}$ and taking account of (3. 35), we find

(3. 37) $$r_j=p_tG_j^{\ t}.$$

Transvecting (3. 37) with $G_k^{\ j}$, we find

(3. 38) $$p_j=-r_tG_j^{\ t}.$$

Substituting (3. 20), (3. 31) and (3. 33) into (3. 14), we find

(3. 39) $$h_s^{\ t}V^sG_t^{\ h}u_j+F_j^{\ t}h_t^{\ h}-h_j^{\ t}F_t^{\ h}=0.$$

Substituting (3. 17), (3. 19) and (3. 33) into (3. 6), we find

(3. 40) $$G_j^{\ t}h_t^{\ s}v_s=0.$$

Substituting (3. 40) into (3. 39), we have

(3. 41) $$F_j^{\ t}h_t^{\ h}-h_j^{\ t}F_t^{\ h}=0.$$

Substituting (3. 40) into (3. 32), we find

(3. 42) $$F_j^{\ t}h_t^{\ s}u_s=0.$$

Substituting (3. 20) into (3. 2), we find

(3. 43) $$G_j^{\ t}h_t^{\ s}u_s=h_j^{\ t}w_t.$$

Substituting (3. 42) and (3. 43) into (3. 5), we have

(3. 44) $$G_j^{\ t}h_t^{\ h}-h_j^{\ t}G_t^{\ h}=0.$$

Substituting (3. 31), (3. 35) and (3. 41) into (2. 1), we obtain

$$[F, F]_{ji}^{\ \ h}+(\nabla_ju_i-\nabla_iu_j)U^h=0.$$

Substitituting (3. 37), (3. 38) and (3. 44) into (2. 2), we obtain

$$[G, G]_{ji}^{\ \ h}+(\nabla_jv_i-\nabla_iv_j)V^h=0.$$

Thus we have the following

THEOREM 3.1. *On a hypersurface of a quaternionic Kaehlerian manifold* $'M^{4n}(n\geqq 2)$, *the condition*

$$[F,F]+du\otimes U=0 \quad and \quad [G,G]+dv\otimes V=0$$

and the condition

$$[F,G]+du\otimes V+dv\otimes U=0$$

are equivalent, where (F,U,u), (G,V,v) *and* (H,W,w) *are the almost contact three structures induced on the hypersurface.*

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