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EXTENSIVE SUBCATEGORIES

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Dedicated to Professor Chi Young Kim on his 60th birthday

1. Introduction.

Using limit-operators, we have established in [10] a method to construct new extensive subcategories from well known extensive subcategories in various subcategories of the category **Haus** of Hausdorff spaces and continuous maps and the category **HUnif** of Hausdorff uniform spaces and uniformly continuous maps. In this vein, the following question is natural: Can every extensive subcategory containing the known extensive subcategory of the category, be constructed with some limit-operator?

For this question, we introduce new operators which satisfy weaker conditions than limit-operators and still give us same machinery as limit-operators. Moreover, for a hereditary subcategory $\boldsymbol{A}$ of **Haus** or **HUnif** and for an extensive subcategory $\boldsymbol{B}$ of $\boldsymbol{A}$, every reflective subcategory of $\boldsymbol{A}$ containing $\boldsymbol{B}$ can be characterized with such an operator on $\boldsymbol{B}$. Also we establish some interesting relationships between those operators and extensive subcategories.

All topological and categorical concepts will be used in the sense of N. Bourbaki [3] and H. Herrlich [7], respectively. In particular, we assume throughout this paper that a subcategory of a category is full and isomorphism-closed.

2. Extensive subcategories.

The category of topological (uniform) spaces and (uniformly resp.) continuous maps will be denoted by **Top** (**Unif**, resp.).

2.1 DEFINITION. Let $\boldsymbol{B}$ be a subcategory of **Top** or **Unif**. An operator l which associates every pair (X, A), where X is an object of $\boldsymbol{B}$ and A is a subset of X, a subset $l_X A$ of X is said to be an *extensive operator* on $\boldsymbol{B}$ if l satisfies the following conditions:

1) if A is a subset of X, then $A \subseteq l_X A \subseteq \mathrm{cl}_X A$, where cl_X is the closure operator on X.

2) if f: $X \longrightarrow Y$ is a morphism in $\boldsymbol{B}$ and A is a subset of X, then $f(l_X A) \subseteq l_Y f(A)$.

3) if A and B are subsets of X with $A \subseteq B$, then $l_X A \subseteq l_X B$. An extensive operator l on $\boldsymbol{B}$ is said to be *idempotent*, if l satisfies the following:

4) if A is a subset of $X \in \boldsymbol{B}$, then $l_X(l_X A) = l_X A$.

It is obvious that every (idempotent) limit-operator (see [8]) is an (idempotent, resp.) extensive operator on **Top**.

2.2 DEFINITION. Let l be an extensive operator on $\boldsymbol{B}$. A subset A of an object X of

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$\boldsymbol{B}$ is said to be *l-closed* if $l_X A = A$. We will denote the family of l-closed subsets of X by $S_l(X)$.

2.3. For any subcategory $\boldsymbol{B}$ of **Top** or **Unif**, let $E = E(\boldsymbol{B})$ be the class of all extensive operators on $\boldsymbol{B}$. We define a relation $\leqq$ on E as follows: for any pair $(l, l') \in E^2$, $l \leqq l'$ iff $l'_X A \subseteq l_X A$ for every $X \in \boldsymbol{B}$ and every subset A of X. Then it is easy to show that $(E, \leqslant)$ becomes a complete "lattice", where $l^0 (l^1)$ with $l^0_X A = \mathrm{cl}_X A$ ($l^1_X A = A$, resp.) is the smallest (largest, resp.) element of E and for any subfamily E' of E, $(\bigvee \{l \mid l \in E'\})_X A = \cap \{l_X A \mid l \in E'\}$ defines the join of E' and $(\bigwedge \{l \mid l \in E'\})_X A = \cup \{l_X A \mid l \in E'\}$ defines the meet of E'.

For the restricted relation of $\leqslant$ on the class of $IE(\boldsymbol{B})$ of all idempotent extensive operators on $\boldsymbol{B}$ again denoted by $\leqslant$, $(IE(\boldsymbol{B}), \leqslant)$ is also a complete "lattice" with the same largest and smallest elements, while the join of a subfamily E' of $IE(\boldsymbol{B})$ in $IE(\boldsymbol{B})$ is the same as that of E' in $E(\boldsymbol{B})$ but the meet of E' in $IE(\boldsymbol{B})$ is not the same as that of E' in $E(\boldsymbol{B})$ (see the following remark).

2.4 REMARK. 1) For any extensive operator l on $\boldsymbol{B}$ there is an associated idempotent extensive operator $\hat{l}$ on $\boldsymbol{B}$ with $S_{\hat{l}}(X) = S_l(X)$ for every $X \in \boldsymbol{B}$, where $\hat{l}_X A = \cap \{B \mid A \subseteq B \text{ and } B \in S_l(X)\}$.

2) The associated idempotent extensive operator $\hat{l}$ on $\boldsymbol{B}$ of an extensive operator l on $\boldsymbol{B}$ turns out to be the largest idempotent extensive operator on $\boldsymbol{B}$ with $l' \leqslant l$ and the meet of a subfamily of $IE(\boldsymbol{B})$ in $IE(\boldsymbol{B})$ is precisely the associated idempotent extensive operator of the meet of the subfamily in $E(\boldsymbol{B})$.

3) For any extensive operator l on $\boldsymbol{B}$, there is an associated idempotent limit-operator $\bar{l}$ on $\boldsymbol{B}$, where for any $X \in \boldsymbol{B}$, $\bar{l}_X$ is defined as the closure operator on X with respect to the topology with $S_l(X)$ as a subbase for the closed sets. Furthermore, for any extensive operator l on **Top**, let $\mathscr{C}(l) = \{X \in \textbf{Top} \mid$ every member of $S_l(X)$ is closed in $X\}$. Then it is obvious that $\mathscr{C}(l) = \mathscr{C}(\bar{l})$. Hence every extensive operator on **Top** generates a coreflective subcategory of **Top**, for $\mathscr{C}(\bar{l})$ is a coreflective subcategory of **Top**(see[8]).

2.5 DEFINITION. Let $\boldsymbol{A}$ be a subcategory of the category **Haus** or **HUnif**. A subcategory $\boldsymbol{B}$ of $\boldsymbol{A}$ is called *extensive* if it is a reflective subcategory of $\boldsymbol{A}$ such that the $\boldsymbol{B}$-reflection maps $r_X \colon X \longrightarrow rX$ are dense embeddings for each $X \in \boldsymbol{A}$.

It is well known that for every epi-reflective subcategory $\boldsymbol{B}$ of **Haus**, there is an epi-reflective subcategory $\boldsymbol{RB}$ of **Haus** such that $\boldsymbol{B}$ is extensive in $\boldsymbol{RB}$ and for any X in **Haus**, the $\boldsymbol{B}$-reflection of X is factorized through the $\boldsymbol{RB}$-reflection of X and $\boldsymbol{B}$-reflection of the $\boldsymbol{RB}$-reflection of X. Hence every epi-reflective subcategory of **Haus** can be completely determined by a certain extensive subcategory in a (hereditary) subcategory of **Haus** (see [6]).

Let $\boldsymbol{B}$ be an extensive subcategory $\boldsymbol{A}$ of **Haus** or **HUnif**. For an idempotent extensive operator l on $\boldsymbol{B}$, let $\boldsymbol{B}_l$ be the subcategory of $\boldsymbol{A}$ determined by those objects of $\boldsymbol{A}$ which are l-closed in their $\boldsymbol{B}$-reflection spaces.

2.6 THEOREM. *If $\boldsymbol{A}$ is hereditary, then $\boldsymbol{B}_l$ is also an extensive subcategory of $\boldsymbol{A}$.*

Proof. For every $X \in \boldsymbol{A}$, let $r_X \colon X \longrightarrow rX$ be the $\boldsymbol{B}$-reflection of X such that X is a dense subspace of rX and r_X is the natural embedding. Let r_lX be the subspace of rX with $l_{rX}X$ as its underlying set. Since $\boldsymbol{A}$ is hereditary, r_lX belongs to $\boldsymbol{A}$ and the natural embedding $r_lX \longrightarrow rX$ is a $\boldsymbol{B}$-reflection of r_lX. Hence r_lX is l-closed in its $\boldsymbol{B}$-reflection space rX, so that r_lX belongs to $\boldsymbol{B}_l$. By the exactly same arguments as those in [10], we can conclude that the natural embedding $X \longrightarrow r_lX$ is a $\boldsymbol{B}_l$-reflection of X.

2.7 REMARK. 1) The correspondence $l \longrightarrow \boldsymbol{B}_l$ between $(IE(\boldsymbol{B}), \leqslant)$ and the class $\mathrm{Ext}_{\boldsymbol{B}}\boldsymbol{A}$ of all extensive subcategories of $\boldsymbol{A}$ containing $\boldsymbol{B}$ with the inclusion relation is monotone.

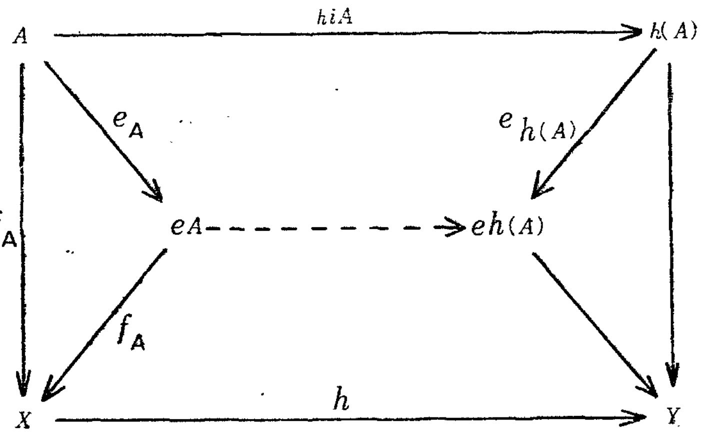
2) It is well known [2], [9], that for any subspace Y of the Katětov extension κX (see [11]) of a Hausdorff space X, κX and κY are homeomorphic if Y contains X. With this and the same argument as that in the above theorem, it is easy to show that the above theorem holds for the case of $\boldsymbol{A}=$**pHaus** (see [5]) and $\boldsymbol{B}=$the subcategory of **pHaus** determined by all H-closed spaces, i.e. for any idempotent extensive operator l on the subcategory $\boldsymbol{H}$ of **pHaus** determined by all H-closed spaces, the subcategory $\boldsymbol{H}_l$ of **pHaus** determined by spaces which are l-closed in their Katětov extensions is also extensive in **pHaus**.

3. Reflective subcategories of a Hereditary subcategory of Haus or HUnif.

For a subcategory $\boldsymbol{A}$ of **Haus** or **HUnif** and an extensive subcategory $\boldsymbol{B}$ of $\boldsymbol{A}$, every H-closed $\boldsymbol{A}$-object belongs to $\boldsymbol{B}$ so that one can easily guess the smallest extensive subcategory of $\boldsymbol{A}$. Furthermore, it is easy to show that every reflective subcategory of $\boldsymbol{A}$ containing $\boldsymbol{B}$ is also extensive in $\boldsymbol{A}$.

3.1 THEOREM. *Let $\boldsymbol{A}$ be a hereditary subcategory of* **Haus** *or* **HUnif** *and $\boldsymbol{B}$ an extensive subcategory of $\boldsymbol{A}$. For any reflective subcategory $\boldsymbol{E}$ of $\boldsymbol{A}$ containing $\boldsymbol{B}$, there exists an idempotent extensive operator $l^{\boldsymbol{E}}$ on $\boldsymbol{B}$ with $\boldsymbol{E}=\boldsymbol{B}_{l^{\boldsymbol{E}}}$.*

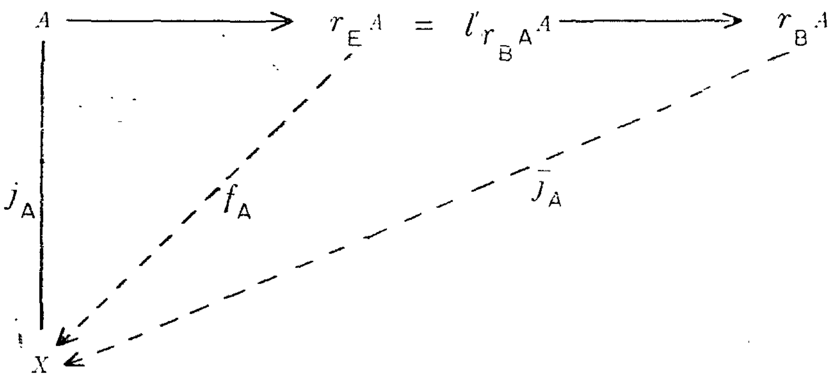
Proof. By the above remark, $\boldsymbol{E}$ is also extensive in $\boldsymbol{A}$. For any X in $\boldsymbol{A}$, let e_X: $X \longrightarrow eX$ be an $\boldsymbol{E}$-reflection of X. Let A be a subset of an object X of $\boldsymbol{B}$. Since $\boldsymbol{A}$ is hereditary, the subspace A of X belongs to $\boldsymbol{A}$. For the natural embedding j_A: $A \longrightarrow X$, there is a unique morphism f_A: $eA \longrightarrow X$ in $\boldsymbol{E}$ with $f_Ae_A=j_A$, for $X\in\boldsymbol{B}\subseteq\boldsymbol{E}$. We define l_XA by $f_A(eA)$. We wish to show that the operator l defined as the above is an extensive operator on $\boldsymbol{B}$. Firstly, $A=j_A(A)=f_Ae_A(A)\subseteq f_A(eA)=l_XA$, i.e. $A\subseteq l_XA$. Moreover, $l_XA=f_A(eA)=f_A(\mathrm{cl}_{eA}(e_A(A)))\subseteq \mathrm{cl}_X f_A(e_A(A))=\mathrm{cl}_X j_A(A)=\mathrm{cl}_XA$, i.e. $l_XA\subseteq \mathrm{cl}_XA$. Secondly, for any morphism h: $X \longrightarrow Z$ in $\boldsymbol{B}$, we have the following diagram (Shown in the end of the proof) in which the outer rectangle and the upper trapezoid commute, where $j_{h(A)}$, $e_{h(A)}$ and $f_{h(A)}$ can be understood such as j_A, e_A and f_A, and $\bar{h}$ is the unique morphism determined by e_A and $e_{h(A)}h|A$. Since $f_{h(A)}\bar{h}e_A=f_{h(A)}e_{h(A)}h|A=j_{h(A)}h|A=hj_A$ $=hf_Ae_A$, $f_{h(A)}\bar{h}=hf_A$, for e_A is the reflection map. Hence $h(l_XA)=h(f_A(eA))=f_{h(A)}\bar{h}(eA)$ $\subseteq f_{h(A)}(eh(A))=l_Yh(A)$, i.e. $h(l_XA)\subseteq l_Yh(A)$. For the condition 3), the proof is simple and left to the reader. Furthermore, for any $X\in\boldsymbol{B}$, the family $S_l(X)$ of l-closed subsets of X is precisely the family of subsets of X which belong to $\boldsymbol{E}$ as subspaces of X. Indeed, suppose a subspace A of $X\in\boldsymbol{B}$ does not belong to $\boldsymbol{E}$. Since $\boldsymbol{E}$ is extensive in $\boldsymbol{A}$, e_A is not onto and A is dense in eA. Hence $\phi\neq f_A(eA-e_A(A))\subseteq f_A(eA)-f_A(A)$ $=l_XA-A$; A is not l-closed. It is very simple to show that the other inclusion and the proof is left to the reader. Let $l^{\mathbf{E}}$ be the associated idempotent extensive operator on $\mathbf{B}$ with the above extensive operator l. Since $S_l(X)=S_{l^{\mathbf{E}}}(X)=\{A\subseteq X \mid A\in\mathbf{E}\}$, it is obvious that $\mathbf{B}_{l^{\mathbf{E}}}=\mathbf{E}$. This completes the proof.



In what follows, let $\mathbf{A}$ and $\mathbf{B}$ be the same categories as those in Theorem 3.1.

3.2 Remark. For any $\mathbf{E}\in \mathrm{Ext}_{\mathbf{B}}\mathbf{A}$, $l^{\mathbf{E}}$ is the largest element of $IE(\mathbf{B})$ with $\mathbf{E}=\mathbf{B}_l$.

Proof. Let l' be an element of $IE(\mathbf{B})$ with $\mathbf{B}_{l'}=\mathbf{E}$. For any $X\in\mathbf{B}$ and any subset A of X, we have the following commutative diagram



where $r_{\mathbf{B}}A$ is the $\mathbf{B}$-reflection of A, j_A and f_A are defined as in Theorem 3.1, and $\bar{j}_A$ is determined by j_A and the reflection $A \longrightarrow r_{\mathbf{B}}A$. Then $l_XA=f_A(l'r_{\mathbf{B}A}A)\subseteq l'_Xf_A(A)=l'_XA$, where l is the extensive operator constructed in Theorem 3.1. Hence $l'\leqslant l$. By the Remark 2.4, $l'\leqslant l^{\mathbf{E}}$.

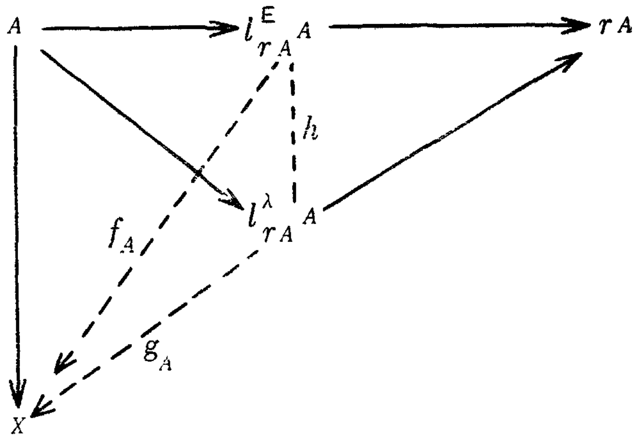
3.3 COROLLARY. *The correspondence* $\boldsymbol{E} \longrightarrow l^{\boldsymbol{E}}$ *between* $Ext_{\boldsymbol{B}}\boldsymbol{A}$ *and* $IE(\boldsymbol{B})$ *is one-one but not necessarily onto.*

Proof. The first assertion is an immediate consequence of Theorem 3.1. For the second part, let $\boldsymbol{A}$ be the category of completely regular spaces and continuous maps, $\boldsymbol{B}$ the category **Comp** of compact spaces and continuous maps, and $\boldsymbol{E}$ the category **RComp** of real compact spaces and continuous maps. S. Mrówka has shown [13] that there is a completely regular space M which can be represented as the union of two closed subsets A, B such that each of them is realcompact in its relative topology and which is not realcompact. Hence $l^{\boldsymbol{E}}_{\beta M}(A\cap B) \neq l^{\boldsymbol{E}}_{\beta M}A \cap l^{\boldsymbol{E}}_{\beta M}A$, where βM is the Stone-Čech compactification of M. Let $\mathrm{cl}_{\aleph_1}$ be the Q-closure operator (see [12]). Then it is well known [12] that **RComp**= **Comp** $\mathrm{cl}_{\aleph_1}$ and $\mathrm{cl}_{\aleph_1}(A\cup B)=\mathrm{cl}_{\aleph_1}A\cup \mathrm{cl}_{\aleph_1}B$. Hence $\mathrm{cl}_{\aleph_1}\neq l^{\boldsymbol{E}}$.

3.4 REMARK. By the above example, we can also conclude that $\boldsymbol{B}_{\bar{l}}$ may contain $\boldsymbol{E}$ properly, where $\bar{l}$ is the associated idempotent limit-operator on $\boldsymbol{B}$ with l.

3.5 THEOREM. *The correspondence* $l \longrightarrow \boldsymbol{B}_l$ *between* $(IE(\boldsymbol{B}), \leqslant)$ *and* $Ext_{\boldsymbol{B}}\boldsymbol{A}$ *with the inclusion relation preserves arbitrary joins and meets.*

Proof. Let $(l^\lambda)_{\lambda\in\Lambda}$ be a subfamily of $IE(\boldsymbol{B})$. Regarding joins, let $s=\bigvee\{l^\lambda|\lambda\in\Lambda\}$. By the remark 2.7, $\boldsymbol{B}_s$ is an upper bound of $\{\boldsymbol{B}_{l^\lambda}|\lambda\in\Lambda\}$. For any upper bound $\boldsymbol{E}$ of $\{\boldsymbol{B}_{l^\lambda}\}$ in $\mathrm{Ext}_{\boldsymbol{B}}\boldsymbol{A}$, there is an idempotent extensive operator $l^{\boldsymbol{E}}$ on $\boldsymbol{B}$ with $\boldsymbol{E}=\boldsymbol{B}_{l^{\boldsymbol{E}}}$. For any subset A of an object X of $\boldsymbol{B}$ and for any $\lambda\in\Lambda$, we have the following commutative diagram



where rA, $l^{\boldsymbol{E}}_{rA}A$ and $l^\lambda_{rA}A$ are reflections of A with respect to $\boldsymbol{B}$, $\boldsymbol{E}$ and $\boldsymbol{B}_{l^\lambda}$ respectively (also see Theorem 2.6) and f_A, g_A and h are determined by the reflection property and $\boldsymbol{B}\subseteq\boldsymbol{B}_{l^\lambda}\subseteq\boldsymbol{E}$. By the definition of l in Theorem 3.1, $l_XA=f_A(l^{\boldsymbol{E}}_{rA}A)=g_Ah(l^{\boldsymbol{E}}_{rA}A)\subseteq g_A(l^\lambda_{rA}A)\subseteq l^\lambda_X g_A(A)=l^\lambda_XA$; $l^\lambda\leqslant l$. By the Remark 2.4, $l^\lambda\leqslant l^{\boldsymbol{E}}$. Hence $s\leqslant l^{\boldsymbol{E}}$, i.e. $\boldsymbol{B}_s\subseteq B_{l^E}=\boldsymbol{E}$. Hence $\boldsymbol{B}_s$ is the least upper bound of $\{\boldsymbol{B}_{l^\lambda}\}$.

Regarding meets, let $m=\bigwedge\{l^\lambda|\lambda\in\Lambda\}$. It is obvious that $\boldsymbol{B}_m$ is a lower bound of

$\{B_{l\lambda} \mid \lambda \in \Lambda\}$. Let E be a lower bound of $\{B_{l\lambda} \mid \lambda \in \Lambda\}$. Since $E \subseteq B_{l\lambda}$ for any $\lambda \in \Lambda$, $l^{\lambda}_{rX}X = X$ for every $X \in E$; $m_{rX}X = X$ (see 2.3). Hence X belongs to B_m. This completes the proof.

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