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CRITERIA OF METRIZABILITY

BY KYUNG BAI LEE

There are several conditions imposed on a symmetric for a space sufficient to imply the metrizability of the space. In this note we will investigate how Niemyzki-Wilson's condition (Π), Arhangel'skii-Martin's condition (A) and Harley III & Fulkner's condition (HF) work on o-metrics or on strong o-metrics. When an o-metric for a space has one of these conditions, the space turns out to be a very familiar one to us; for example, a developable, a g-developable or a Nagata space. Correlation between these conditions proven in this note makes us derive easily the three metrization theorems mentioned above, from Bing's theorem all together.

Let X be a topological space and d a nonnegative real-valued function defined on $X\times X$ such that $d(x,y)=0$ if and only if $x=y$. Such a function d is called an *o-metric*(15) for X provided that a subset U of X is open if and only if $d(x, X\text{-}U)>0$ for each $x\in U$, An o-metric d is called a *strong o-metric* [15] if each sphere $S(x;r)=\{y\in X\colon d(x,y)<r\}$ is a neighborhood of x; a *symmetric* if $d(x,y)=d(y,x)$ for each x and y; a *semi-metric* if d is a symmetric such that $x\in M$ if and only if $d(x,M)=0$.

A *development* for a space X is a sequence $\gamma=(\gamma_1,\gamma_2,\gamma_3,\cdots)$ of open covers of X such that γ_{n+1} refines γ_n and [2] $\{\mathrm{st}(x,\gamma_n):n\in N\}$ forms a local base at x. A space is *developable* if it has a development. Bing[3] proved that a space is metrizable if and only if it is paracompact and developable. Let g be a map from $N\times X$ to 2^X. We call g a first countable *COC-map* (=countable open covering map) if $\mathcal{W}_x=\{g(n,x)\colon n\in N\}$ forms a local base at x, and a g-first countable *CWC*-map (=countable weakly-open covering map) if $\mathcal{W}_x$ forms a weak-base [2]. Heath [10] characterized developable spaces by existence of first countable *COC-g* with an additional condition that x, $x_n\in g(n,y_n)$ for each $n\in N$ implies that the sequence $\{x_n\}$ converges to x. Similarly, a space is said to be *g-developable* (12) if it has a g-first counable *CWC*-map g such that x, $x_n\in g(n,y_n)$ for each $n\in N$ implies that the sequence $\{x_n\}$ converges to x. A Hausdorff space is developable (g-developable) if and only if it is semimetrizable(symmetrizable) via a semi-metric(symmetric, respectively) under which all convergent sequences are Cauchy (see [5] and [12] for details). In [13], it is shown that, among Hausdorrff spaces, g-developable spaces are precisely almost weakly-open Π-images of metric spaces.

A *Nagata* space[6] is a T_1-space such that for each $x\in X$, there exist sequences of neighborhoods of x, $\{f(n,x)\colon n\in N\}$ and $\{g(n,x)\colon n\in N\}$ such that [1] for each x, $\{f(n,x)\colon n\in N\}$ is a local base at x, and (2) for all x and y in X, $g(n,x)\cup g(n,y)\neq\phi$ implies $x\in f(n,y)$. Ceder[6] has shown that Nagata spaces are precisely first countable stratifiable spaces. On the other hand, Heath[10] characterized Nagata spaces by first countable *COC*-map: A space is Nagata if and only if it has a first countable *COC*-map g such that $g(n,x)\cup g(n,x_n)\neq\phi$ implies that the sequence $\{x_n\}$ converges to x.

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This note is motivated from the following metrization theorems which tell us some appropriate conditions imposed on a symmetric for a space guarantee the metrizability of the space.

THEOREM. *The following are equivalent to the metrizability of a space X.*

1 (*Niemyzki–Wilson*). *X is regular, symmetrizable via d such that, whenever* $\lim_n d(y_n, x) = \lim_n d(x_n, y_n) = 0$ *then* $\lim_n d(x_n, x) = 0$.

2 (*Arhangel'skii–Martin*). *X is symmetrizable via d satisfying* $d(F, K) > 0$ *whenever F is closed, K compact, and* $F \cap K = \phi$.

3 (*Harley* III *& Faulkner*). *X is symmetrizable via d such that for each closed set F and each* $x \in X-F$ *there exists* $\varepsilon > 0$ *such that* $S(x;\varepsilon) \cap S(F;\varepsilon) = \phi$.

Consider the following conditions on an o-metric for a space:

(AN) For each $x \in X$ and for any $\varepsilon > 0$ there exists a $\delta = \delta(x, \varepsilon) > 0$ such that $y \in S(x;\delta)$ and $z \in S(x;\delta)$ implies $d(y, z) < \varepsilon$.

(Π_1) From $\lim_n d(x, y_n) = \lim_n d(x_n, y_n) = 0$ it follows that $\lim_n d(x, x_n) = 0$.

(Π_2) From $\lim_n d(y_n, x) = \lim_n d(y_n, x_n) = 0$ it follows that $\lim_n d(x, x_n) = 0$.

(A) If $F \cap K = \phi$, where F is closed, K compact, then $d(F, K) > 0$.

(HF_1) For each closed set F and each $x \in X-F$, there exists $\varepsilon > 0$ such that $S(x;\varepsilon) \cap S(F;\varepsilon) = \phi$.

(HF_2) For each closed set F and each $x \in X-F$, there exists $\varepsilon > 0$ such that $S^*(x;\varepsilon) \cap S^*(F;\varepsilon) = \phi$, where $S^*(x;r) = \{y \in X: d(y, x) < r\}$.

The reader can find (AN) in (1), (Π) in (16), (A) in (2) and (HF) in (8). In what follows spaces will mean only Hausdorff spaces.

LEMMA 1. *In a (Hausdorff) space X o-metrizable via an o-metric d,* $\lim_n d(x, x_n) = 0$ *if and only if the sequence* $\{x_n\}$ *converges to x.*

Proof. Let $\{x_n\}$ converge to x. Assume there exists an $\varepsilon > 0$ and a subsequence $\{x_{ni}: i \in N\}$ of $\{x_n\}$ such that $d(x, x_{ni}) \geqq \varepsilon$ for all $i \in N$. Let $F = \{x, x_{n1}, x_{n2}, \cdots\}$. For any $y \in X-F$, there is a $\delta > 0$ such that $S(y;\delta) \subset X-F$ since F is closed. For $x, S(x;\varepsilon) \subset (X-F) \cup \{x\}$. These imply that $F - \{x\}$ is also closed, a contradiction. The converse is obvious.

LEMMA 2. *Let* $\{x_n\}$ *be a sequence in a space X such that any subsequence* $\{x_{ni}: i \in N\}$ *of* $\{x_n\}$ *has a subsequence convergent to some fixed point x. Then the sequence* $\{x_n\}$ *converges to x.*

PROPOSITION 1. *Let X be a space o-metrizable via d. The conditions* (Π_2) *and* (HF_2) *on d are equivalent.*

Proof. Let d satisfy (Π_2). Assume there exist a closed set F and a point $x \in X-F$ such that $S^*(x;1/n) \cap S^*(F;1/n) \neq \phi$ for each $n \in N$. Then there are sequences $\{y_n\}$ in

X and $\{x_n\}$ in F such that $d(y_n, x) < 1/n$ and $d(y_n, x_n) < 1/n$ for each $n \in N$. This implies that the sequence $\{x_n\}$ converges to x, a contradiction.

Conversely let d satisfy (HF_2). Suppose

$$\lim_n d(y_n, x) = \lim_n d(y_n, y_n) = 0.$$

To show $\{x_n\} \to x$, we may assume $x_n \neq x$ for any $n \in N$. Let $F = \mathrm{cl}\{x_n : n \in N\}$. By the hypothesis, $S^*(x; 1/n) \cap S^*(F; 1/n) \neq \phi$ for any $n \in N$. The condition (HF_2) ensures that $x \in F$, and hence that the sequence $\{x_n\}$ has a subsequence convergent to x. Passing to subsequences, we have the result.

THEOREM 1. *The following are equivalent*:

(1) *X is g-developable,*

(2) *X is symmetrizable via a symmetric satisfying (AN),*

(3) *X is o-metrizable via an o-metric satisfying (AN), and*

(4) *X is o-metrizable via an o-metric satifyjng (Π_2) (or equivalently, (HF_2)).*

Proof. Lee (12) has shown ($1 \Leftrightarrow 2$). ($2 \Rightarrow 3$) is obvious.

($3 \Rightarrow 4$). Let d be a compatible o-metric satisfying(AN), We can determine a positive real-valued function δ on $X \times N$ such that diam $S(x : \delta\ x, n)) < 1/n$ and that $\delta(x, n+1) < \delta(x, n)$. Define a new o-metric b by

$$b(x, y) = 1/\inf\{j \in N :\ d(x, y) \geqslant \delta\ (x, j)\}$$

Note that $b(x, y) < 1/n$ if and only if $d(x, y) < \delta(x, n)$. Hence $S_b(x; 1/n) = S_d(x; \delta(x, n))$. This implies that b is a compatible o-metric for X. To show b satisfies (Π_2), let

$$\lim_n b(y_n, x) = \lim_n b(y_n, x_n) = 0.$$

Then there exists a subsequence $\{n_i : i \in N\}$ of N such that

$$b(y_{ni}, x) < 1/i \text{ and } b(y_{ni}, x_{ni}) < 1/i$$

for each $i \in N$. By the definition of b, $d(y_{ni}, x) < \delta(y_{ni}, i)$ and $d(y_{ni}, x_{ni}) < \delta(y_{ni}, i)$, hence $x,\ x_{ni} \in S_d(y_{ni},\ \delta(y_{ni}, i))$, which implies $d(x, x_{ni}) < 1/i$. Thus we have a subsequence $\{x_{ni} : i \in N\}$ of $\{x_n\}$ convergent to x. Passing to subsequences, we have the result.

($4 \Rightarrow 1$). Let d be a compatible o-metric for X satisfying (Π_2). We define a g-first countable CWC-map g by $g(n, x) = S(x; 1/n)$. Let $x,\ x_n \in g(n, y_n)$. This implies

$$d(y_n, x) < 1/n \text{ and } d(y_n, x_n) < 1/n.$$

Now the condition (Π_2) tells us that the sequence $\{x_n\}$ converges to x.

Note that the condition (AN) and (Π_2) on o-metrics are not equivalent. If they are equivalent, a space symmetrizable via a symmetric satisfying (AN) would be metrizable (See Theorem 3 below), which is false. Also note that semimetrizability of a space via a semimetric satisfying (AN) is not equivalent to g-developability of that space. In fact, we have the following

THEOREM 1′. *The following are equivalent:*

(1) *X is developable,*

(2) *X is semimetrizable via a semimetric satisfying (AN), and*

(3) *X is o-metrizable via a strong o-metric satisfying* (Π_2) *(or equivalently,* (HF_2)*).*

Proof. (1⇒2). See Morton Brown [5].

(2⇒3). In the proof of Theorem 1 (3⇒4), note that the new o-metric b is actually a strong one if d is strong.

(3⇒1). See Nedev (16), Lemma 14.

Let us give our attention to another direction. The equivalence of (Π_1) and (A) on o-metries was noted by Nedev [16] whose proof is given in the following

PROPOSITION 2. *Let X be a space o-metrizable via d. The conditions* (Π_1), (HF_1) *and (A) on d are equivalent.*

Proof. (Π_1⇒HF_1). See the corresponding part of Proposition 1.

(HF_1⇒A). Assume there exist a closed F and a compact K such that $F\cap K=\phi$ and $d(F,K)=0$. Then there exists a sequence $\{x_n\}$ in K such that $d(F,y_n)<1/n$ for each $n\in N$. Since K is compact, there exist a subsequence $\{y_{ni}\}$ of $\{y_r\}$ and a point $x\in K$ such that $d(x,y_{ni})<1/i$ for each $i\in N$. Now $S(x;1/i)\cap S(F;1/i)\in y_{ni}$ for each $i\in N$ implies that $x\in F$ a contradiction.

(A⇒Π_1). Assume $\lim d(x,y_n)=\lim d(x_n,y_n)=0$ but $\lim d(x,x_n)\neq\phi$. There exist a subsequence $\{x_{ni}\}$ of $\{x_n\}$ and a closed set F not containing x but containing all x_{ni}. Let K be a compact set $\{x,y_k,y_{k+1},\cdots\}$ such that $K\cap F=\phi$. But $d(F,K)\leqslant\inf d(x_{ni},y_{ni})=0$, a contradiction.

THEOREM 2. *The following are equivalent:*

(1) *X has a g-first countable CWC-map g such that* $g(n,x)\cap g(n,x_n)\neq\phi$ *for each* $n\in N$ *implies that the sequence* $\{x_n\}$ *converges to x,*

(2) *X is o-metrizable via an o-metric satisfying* (Π_1) *or equivalently,* (HF_1) *or (A).*

Proof. (1⇒2). Define an o-metric d for X by

$$d(x,y)=1/\inf\{j\in N:\ y\notin g(j,x)\}.$$

Suppose $\lim_n d(x,y_n)=\lim_n d(x_n,y_n)=0$. There exist subsequecences $\{x_{ni}\}$, $\{y_{ni}\}$ of $\{x_n\}$, $\{y_n\}$, respectively, such that $d(x,y_{ni})<1/i$ and $d(x_{ni},y_{ni})<1/i$, or equivalently, $y\in g(i,x)\cap g(i,x_{ni})$ for each $i\in N$. By hypothesis, $\{x_{ni}\}\to x$. Passing to subsequences, the sequence $\{x_n\}$ converges to x.

(2⇒1). Modify the corresponding part of Theorem 1.

THEOREM 2′. *The following are equivalent:*

(1) *X is a Nagata space, and*

(2) *X is o-metrizable via a strong o-metric satisfying* (Π_1)*, or equivalently,* (HF_1) *or (A).*

Proof. Note that in the proof of Theorem 2 (1⇒2), the o-metric induced from g is a strong one if g is a first countable COC-map. Conversely, a first countable space with a g-first countable CWC-map satisfying the condition (1) of Theorem 2 is indeed a Nagata space (See (19), Theorem 1.10).

LEMMA 3 (Nedev, (16)). *If a space is symmetrizable via a symmetric satisfying* (Π_1), *the space is first countable.*

Now we are able to prove the three metrization theorems all together.

THEOREM 3 (*Niemyzki-Wilson, Arhangel'skii-Martin, Herley* III & *Faulkner*). *A space is metrizable if and only if it is symmetrizable via a symmetric satisfying* (Π), *or equivalently* (A) *or* (HF).

Proof. In the case of symmetric, the conditions(Π_1), (Π_2), (HF_1), (HF_2) and (A) on a symmetric are equivalent. By virtue of Lemma 3, such a space is first countable. Note that all o-metrics on a Hausdorff space are equivalent(11). Applying Theorem 1′ and Theorem 2′, we knew such a space is developable and Nagata. Since a paracompact developable space is metrizable(3), the proof is completed.

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Seoul National University