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EXTENSION OF WEAKLY COMPACT OPERATORS

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1. Introduction

J. Lindenstrauss conjectured in [2] that only finite dimensional Banach spaces X have the following "into" extension property of weakly compact operators:
Every weakly compact operator from any Banach space Y into X has a weakly compact extension from any Banach space Z containing Y into X.

He pointed out in [2] that

(*a*) *if a Banach space X has the "into" extension property, then the second dual X^{**} of X is a P space, and that*
(*b*) *if a Banach space X contains a subspace isomorphic to c_0 or to an infinite dimensional reflexive space, then X has no "into" extention property.*

J. Lindenstrauss also conjectured that

every infinite dimensional Banach space contains an infinite dimensional subspace that is either reflexive or isomorphic to c_0 or l_1,

which turned out (cf. [7]) to be equivalent to the fact that

every infinite dimensional Banach space contains an infinite dimensional subspace with the property (u).

In regard of the above conjecture and the result (b), it is natural to ask whether any Banach space containing a subspace isomorphic to l_1 has the "into" extension property. In this paper we shall show, as a first step for the question, that any Banach space containing a complemented isomorph of l_1 fails to have the "into" extension property.

2. Theorems

THEOREM 1. *Any infinite dimensional Banach space X with an unconditional basis has no "into" extension property.*

Proof. It is proved in [5] that any Banach space with an unconditional basis such that X^{**} is a P space is isomorphic to $c_0(\Gamma)$ for a suitable set Γ. Therefore, if X has the "into" extension property, then X^{**} is a P space and hence, since X is infinite dimensional, X contains a subspace isomorphic to c_0, which be contrary to (b).

COROLLARY. *l_1 has no "into" extension property.*

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Proof. Since l_1 has an unconditional basis, the corollary follows directly from the theorem 1. We shall offer another proof. We recall first that if any conjugate space X^* contains a subspace isomorphic to c_0, then X contains a complemented subspace isomorphic to l_1 (cf. [1]). Now if l_1 has the "into" extension property, then $l_1^{**}=l_\infty^*$ is a P space and hence contains a subspace isomorphic to c_0 (cf. [6]). From this it follows that l_∞ contains a subspace isomorphic to l_1 that is complemented in l_∞. But this contradicts to the fact that any infinite dimensional complemented subspace of l_∞ is isomorphic to l_∞ (cf. [3]).

THEOREM 2. *Any Banach space X containing a complemented subspace isomorphic to l_1 has no "into" extension property.*

Proof. Let E be a Banach space containing l_1 as a complemented subspace and be isomorphic to X. Then $E=l_1\oplus F$ for some closed subspace F of E. Since l_1 has no "into" extension property, there exists a weakly compact operator T: $Y\longrightarrow l_1$ which has no extension from a Banach space Z $(Z\supset Y)$ to l_1. Note that if S is the unit ball of Y, then $T(S)$ is relatively weakly compact subset of E since l_1 is weakly closed in E. Therefore $T:Y\longrightarrow E$ is weakly compact. Assume that T has an extension $\tilde{T}$ from the Banach space Z to E. Let π: $E\longrightarrow l_1$ be the natural projection. Then $\pi\circ\tilde{T}$ is an extension of T from Z to l_1, which is a contradiction. This shows that E has no "into" extension property. By transfering E to X by an isomorphism, X has no "into" extension property.

THEOREM 3. *Let X be a Banach space that can be imbedded in a Banach space with an absolute basis. If X contains a subspace isomorphic to l_1, then X has no "into" extension property.*

Proof. In this case X contains a subspace isomorphic to l_1 which is complemented in X (cf. [1] C. 7). Therefore the theorem follows from the theorem 2.

REMARK 1. The P space $C[0,1]$ contains a subspace isomorphic to l_1 but no complemented isomorph of l_1. In fact, $C[0,1]$ has the "into" extension property for compact operators but l_1 has not. (cf. Proof of the theorem 2)

REMARK 2. If X^{**} is an infinite dimensional P space, then X^* contains a complemented subspace isomorphic to the direct sum of finitely many l_1's. But in general, a Banach space whose dual contains a subspace isomorphic to l_1 fails to contain an isomorph of c_0 (cf. [4]).

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