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A NOTE ON L^2-CONTINUITY OF PSEUDODIFFERENTIAL OPERATORS

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1. Introduction. A pseudodifferential operator $A \varepsilon L_{\rho}{}^{m}{}_{,\delta} : S \rightarrow S'$ is defined by

$$Au(x_1) = (2\pi)^{-n} \int e^{i(x_1-x_2)\cdot\xi} a(x_1, x_2, \xi) u(x_2) dx_2 d\xi \tag{1.1}$$

where the symbol $a(x_1, x_2, \xi)$ belongs to the space $S_{\rho}{}^{m}{}_{,\delta}$ (see [5], [6], [7]), namely

$$|\partial_{x_j}{}^{\alpha} \partial_{\xi}{}^{\beta} a(x_1, x_2, \xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m+\delta_j|\alpha|-\rho|\beta|}, \quad j=1,2 \tag{1.2}$$

In particular if $a(x_1, x_2, \xi) = P(\xi)$, a polynomial in ξ, then A coincides with the differential operator $P(D), D = (-i\frac{\partial}{\partial x_1}, \ldots\ldots, -i\frac{\partial}{\partial x_n})$.

Hörmander ([6]) showed that (1.1) is L^2-continuous if $a(x_1, x_2, \xi)$ has a compact support in (x_1, x_2)-variables, satisfies (1.2) for all multiindices α, β, and $m < \{\rho - (1/2)(\delta_1+\delta_2)\}$ where $0 \leq \rho \leq \delta_j < 1$. He also proved that (1.1) fails to be L^2-continuous if $m > n\{\rho - (1/2)(\delta_1+\delta_2),\}$. Later on Calderón and Vaillancourt ([2]) relaxed the order of differentiabilities to a finite number, that is $0 \leq |\alpha| \leq 2m_j$, $0 \leq |\beta| \leq 2[n/2]+2$, m_j is the smallest integer $\geq (5n)/(4(1-\delta_j))$, $j=1,2$, and removed the assumption of compact supportnessin (x_1, x_2)-variables while assuming compact supportness in ξ-variable. They alsosettled down the critical case of $m = n\{\rho - (1/2)(\delta_1+\delta_2)\}$. Cordes ([4]) proved similar results with $0 \leq |\alpha|, |\beta| \leq [n/2]+1$, for a symbol $a(x, \xi)$ of $2n$-variables using Banach algebra techniques. Childs ([3]) improved Cordes' results by assuming instead multiple Hölder continuity of order greater than 1/2. Kato ([8]) also proved similar theorems to Cordes employing Banach algebra techniques and partition of unity. It is known that L^2-continuity of (1.1) fails for the spaces $S^0_{\rho,1}$, $0 < \rho < 1$ (Kumano-Go ([8])) and $S^0_{1,1}$ (Chin-Hung Ching, Ph. D. Thesis, New York University, 1971).

In this paper we further improve Calderón and Vaillancourt's result on the order of differentiabilities with respect to the x_j-variables $(j=1,2)$, and drop the assumption of compact supportness in ξ-variable. In the end we list two open problems related to this paper.

2. Main Theorem. THEOREM. *Let $a(x_1, x_2, \xi)$ be compactly supported in ξ-variable, and satisfy* (1.2) *for* $0 \leq |\alpha| \leq 2m_j$, $0 \leq |B| \leq 2[n/2]+2$, m_j $(j=1,2)$ *is the smallest interger* $\geq n\{1-(1/4)(\delta_1+\delta_2)\}/(1-\delta_j)$. *If* $m \leq n\{\rho - (1/2)(\delta_1+\delta_2)\}$, *then the pseudodifferential operator A defined by* (1.1) *is L^2-continuous.*

Proof. We essentially follow Calderón and Vaillancourt's proof ([2]). It suffices to prove the theorem for $m = n(\rho - (1/2)(\delta_1+\delta_2))$ because the right hand side of (1.2) is decreasing as m decreases. Let $N = [n/2]+1$. Notice first the identity

$$\{1+(1+|\xi|)^{2N\rho}|x_1-x_2|^{2N}\}^{-1}\{1+(1+|\xi|)^{2N\rho}(-\Delta_{\xi})^N\} e^{i(x_1-x_2)\cdot\xi} = e^{i(x_1-x_2)\cdot\xi}$$

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where $\Delta_\xi = \sum_1^n \partial^2/\partial\xi_j^2$ is the Laplacian.

Integration by parts of (1.1) yields

$$\mathrm{Au}(x_1) = (2\pi)^{-n}\int e^{i(x_1-x_2)\cdot\xi}\, b(x_1, x_2, \xi)\, u(x_2)\, dx_2 d\xi$$

where $b(x_1, x_2, \xi) = [1+(-\Delta_\xi)^N(1+|\xi|)^{2N\rho}]\{a(x_1, x_2, \xi)[1+(1+|\xi|)^{2N\rho}|x_1-x_2|^{2N}]^{-1}\}$.

According to [2: pp. 1185–1186], $b(x_1, x_2, \xi)$ is majorized by

$$(2.1) \qquad |\partial_{x_j}{}^\alpha b(x_1, x_2, \xi)| \leq C\|a\|(1+|\xi|)^{N+\delta_j|\alpha|}G((1+|\xi|)^\rho(x_1-x_2))$$

for $0 \leq |\alpha| \leq 2m_j$, m_j is to be determined later, $\|a\| = \inf C_{\alpha\beta}$ in the right hand side of (1.2), and G is an integrable function such that

$$\int G(x)\, dx \leq 1$$

The following lemma was verified by Calderón and Vaillancourt ([1 : pp. 374–378], [2 : p. 1185]):

LEMMA (Calderón and Vaillancourt). *Let A be a bounded operator on a Hilbert space and let $A(x)$ be a weakly-measurable, uniformly bounded operator valued function on a measure space X with measure dx. If*

$$\|A^*(x_1)A(x_2)\| \leq h_1^2(x_1, x_2)$$

$$\|A(x_1)A^*(x_2)\| \leq h_2^2(x_1, x_2)$$

and if

$$\int h_1(x_1, x)h_2(x, x_2)dx$$

is the kernel of a bounded operator on $L^2(X)$ with norm M^2, then

$$\|\int_B A(x)dx\| \leq M$$

where B is any Borel subset of finite measure of X.

To apply the lemma, first observe that the kernel of A is given by

$$A(\xi)u = (2\pi)^{-n}\int e^{i(x_1-x_2)\cdot\xi} b(x_1, x_2, \xi)u(x_2)dx_2$$

It follows that the kernel of $A^*(\xi_1)A(\xi_2)$ is given by

$$(2\pi)^{-n}\int e^{-i(\xi_1-\xi_2)\cdot y + i\xi_1\cdot x_1 - i\xi_2\cdot x_2}\, \bar{b}(y, x_1, \xi_1)b(y, x_2, \xi_2)\, dy$$

Noting the identity $|\xi_2-\xi_1|^{-2m_1}(-\Delta_y)^{m_1} e^{i(\xi_2-\xi_1)\cdot y} = e^{i(\xi_2-\xi_1)\cdot y}$, integration by parts yields

$$(2.2) \qquad (2\pi)^{-n}\int e^{-i(\xi_2-\xi_1)\cdot y + i\xi_1\cdot x_1 - i\xi_2\cdot x_2}|\xi_2-\xi_1|^{2m_1}(-\Delta_y)^{m_1}((\bar{b}(y, x_1, \xi)b(y, x_2, \xi_2))dy$$

In view of (2.1) and Leibniz rule, (2.2) is majorized by

$$C\|a\|^2(1+|\xi_1|^N(1+|\xi_2|)^N(1+|\xi_1|+|\xi_2|)^{2m_1\delta_1}|\xi_1-\xi_2|^{-2m_1}.$$
$$\int G((1+|\xi_1|)^\rho(y-x_1))G((1+|\xi_2|)^\rho(y-x_2))dy$$

Since $\int G(y)dy\leq 1$, change of variables and Schwarz inequality reveal

$$\|A^*(\xi_1)\ A(\xi_2)\|\leq h_1^2(\xi_1,\xi_2)$$

where

$$h_1^2(\xi_1,\xi_2)=C\|a\|^2(1+|\xi_1|)^{N+n\rho}(1+|\xi_2|)^{N+n\rho}[1+\frac{|\xi_1-\xi_2|^{2m_1}}{(1+|\xi_1|+|\xi_2|)^{2m_1\delta_1}}]^{-1}$$

Likewise,

$$\|A(\xi_1)A^*(\xi_2)\|\leq h_2^2(\xi_1,\xi_2)$$

where h_2 is obtained from h_1 by replacing δ_1 and m_1 by δ_2 and m_2 respectively. Set

$$h_j^{(1)}(\xi_1,\xi_2)=(1+|\xi_1|)^{-\frac{n}{4}(\delta_1+\delta_2)}(1+|\xi_2|)^{-\frac{n}{4}(\delta_1+\delta_2)-m_j(1-\delta_j)}\cdot\chi_1(\xi_1,\xi_2)$$

$$h_j^{(2)}(\xi_1,\xi_1)=(1+|\xi_1|)^{-\frac{n}{2}(\delta_1+\delta_2)}[1+\frac{|\xi_1-\xi_2|^{2m_j}}{(1+|\xi_1|+|\xi_2|)^{2m_j\delta_j}}]^{-1/2}\cdot\chi_2(\xi_1,\xi_2)$$

$$h_j^{(3)}(\xi_1,\xi_2)=(1+|\xi_1|)^{-\frac{n}{4}(\delta_1+\delta_2)-m_j(1-\delta_j)}(1+|\xi_2|)^{-\frac{n}{4}(\delta_1+\delta_2)}\cdot\chi_3(\xi_1,\xi_2)$$

where χ_1,χ_2 and χ_3 are the characteristic functions of the sets $\{(\xi_1,\xi_2):|\xi_1|\leq\frac{1}{2}\xi_2\}$, $\{(\xi_1,\xi_2):\frac{1}{2}|\xi_2|\leq|\xi_1|\leq 2|\xi_2|\}$ and $\{(\xi_1,\xi_2):2|\xi_2|\leq|\xi_1|\}$ respectively. It is readily seen that

$$h_j\leq C\|a\|(h_j^{(1)}+h_j^{(2)}+h_j^{(3)})$$

A simple computation shows for $j=1,2,3$

$$\int h_1^{(j)}(\xi_1,\xi_2)d\xi_1\leq C(1+|\xi_2|)^{-\frac{n}{2}(\delta_1+\delta_2)-m_1(1-\delta_1)+n} \tag{2.3}$$

$$\int h_2^{(j)}(\xi_1,\xi_2)(1+|\xi_1|)^{-\frac{n}{2}(\delta_1+\delta_2)-m_1(1-\delta_1)+n}d\xi_1$$
$$\leq C(1+|\xi_2|)^{-n(\delta_1+\delta_2)-m_1(1-\delta_1)-m_2(1-\delta_2)+2n}\leq C \tag{2.4}$$

Similar inequalities are obtained when we integrate against the second variable. In this case we have to replace m_1 and δ_1 by m_2 and δ_2 respectvely. Notice that the integrals (2.3) and (2.4) converge if and only if $m_j\geq\frac{n\{1-(1/4)(\delta_1+\delta_2)\}}{1-\delta_j}(j=1,2)$, which subsequently imply $n(\delta_1+\delta_2)+m_1(1-\delta_1)+m_2(1-\delta_2)-2n>0$, and hence the right hand side of (2.4) is bounded by a constant. Consequently we have from (2.3) and (2.4)

$$\int h_1(\xi_1,\xi)\ h_2(\xi,\xi_2)d\xi_1 d\xi\leq C\|a\|^2$$

$$\int h_1(\xi_1,\xi)\ h_2(\xi,\xi_2)d\xi d\xi_2\leq C\|a\|^2$$

Observe that the cross product terms vanish in the above integrals. It follows from the lemma that $A=\int A(\xi)d\xi$ is L^2-continuous whose norm is bounded by $C\|a\|$. This completes the proof.

REMARK. Our estimate is better than that of Calderón and Vaillancourt's by $\frac{n(1+\delta_1+\delta_2)}{4(1-\delta_j)}$ $(j=1,2)$.

COROLLARY. *The theorem still holds without the assumption that $a(x_1,x_2,\xi)$ has a compact support in ξ-variable.*

Proof. Following [6] and [7], we do a partition of unity as follows: Let $e\in C_0^\infty(R^1)$ be a nonnegative radial function such that $e(r)>0$ for $r<1/2$, $e(r)=0$ for $r>3/4$. Let $f(\xi)=e(|\xi|)$ and define $\varphi_k(\xi)=f(\xi-k)[\sum_0^\infty f(\xi-k)]^{-1}$. Then $\{\varphi_k\}$ is a partition of unity.

Furthermore,

$$|\partial_\xi^\beta\varphi_k(\xi)|\leq C_\beta \tag{2.5}$$

Let $\Psi_k(\xi)=\varphi_k(\xi|\xi|^{-\rho})$. Then $\{\Psi_k\}$ is also a partition of unity and (2.5) implies

$$|\partial_\xi^\beta\Psi_k(\xi)|\leq C_\beta(1+|\xi|)^{-\rho|\beta|} \tag{2.6}$$

Choose a sequence $\{\xi_k\}$, $\xi_k\in\operatorname{supp}\Psi_k$ such that

$$|\xi_k|\geq C_1 k \tag{2.7}$$

$$|\xi-\xi_k|\leq C_2|\xi_k|^\rho,\quad \xi\in\operatorname{supp}\Psi_k. \tag{2.8}$$

Here C_1 and C_2 are independent of k. Now set $a_k(x_1,x_2,\xi)=a(x_1,x_2,\xi)\Psi_k(\xi)$. Then each $a_k(x_1,x_2,\xi)$ has a compact support in ξ. Moreover, (1,2), (2.6) and Leibniz rule show that $a_k(x_1,x_2,\xi)$ belongs to the space $S^m_{\rho,\delta}$. Let A_k be the corresponding operator associated with $a_k(x_1,x_2,\xi)$, namely

$$A_k u(x_1)=(2\pi)^{-n}\int e^{i(x_1-x_2)\cdot\xi}a_k(x_1,x_2,\xi)u(x_2)dx_2d\xi$$

The kernel of A_k is given by

$$A_k(x_1,x_2)=(2\pi)^{-n}\int e^{i(x_1-x_2)\cdot\xi}a_k(x_1,x_2,\xi)d\xi$$

Integration by parts s times we have in view of (1.2), (2.6) and (2.8)

$$|A_k(x_1,x_2)|\leq C_s(1+|x_1-x_2|)^{-s}(1+|\xi_k|)^{m-\rho s+\rho n}$$

By taking $s=n+M$ where M is a positive integer such that $\rho M>1$ for $0<\rho<1$, we obtain

$$\int A_k(x_1,x_2)\ dx_1\leq C(1+|\xi_k|)^{m-\rho M}$$

$$\int A_k(x_1,x_2)\ dx_2\leq C(1+|\xi_k|)^{m-\rho M}$$

It follows that

(2.9) $$\|A_k\| \leq C(1+|\xi_k|)^{m-\rho M}$$

Recalling $m=n\{\rho-(1/2)(\delta_1+\delta_2)\}\leq 0$, and $|\xi_k|^{-1}\leq Ck^{-1}$ from (2.7), we finally have from (2.9)

$$\|A\| \leq \sum_0^\infty \|A_k\| < \infty,$$

thereby completing the proof.

In [10], Fefferman introduced wave packet transform whose kernel is in fact a symbol of $3n$-variables in $S^0_{1/2,1/2}$.

3. Some open problems. In [8] Kato verified L^2-continuity of pseudodifferential operators associated with symbols $a(x, \xi)$ in $S^0_{\rho,\delta}$ of $2n$-variables using Banach algebra techniques and partition of unity. His argument heavily depends on the fact that the convolution $a_* g$ defined by

$$(a_* g)(x, D) = \int a(x, \xi) e^{i\xi\cdot x} e^{-ix\cdot D} g(x, D) e^{ix\cdot D} e^{-i\xi\cdot x} dx d\xi$$

exists as a strong integral for $a(x, \xi) \in L^\infty(R^{2n})$ and $g(x, \xi) = \Psi_{n,t}(x)\Psi_{n,s}(\xi), s, t > n/2$, where $\Psi_{n,t}$ is the fundamental solution of the operator $(1-\Delta)^{t/2}$, Δ is the Laplacian. A natural question arises: Is it possible to extend Kato's results to a symbol $a(x_1, x_2, \xi)$ of $3n$-variables? Essentially the question is equivalent to finding a suitable conditions under which a symbol $a(x_1, x_2, \xi)$ of $3n$-variables reduces to a symbol $a(x, \xi)$ of $2n$-variables.

In [3], Childs proved similar [theorems using multiple Hölder continuity of order greater than 1/2. It is not known that his results still hold for the order of 1/2.

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