J. Korean Math. Soc.
Vol. 13, No. 1, 1976

ON THE CURVATURE TENSOR R_{ijkh} OF C-REDUCIBLE FINSLER SPACES

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In 1972 one of the authors introduced the notion of C-reducibility of Finsler space [5]. We have interesting examples of C-reducible Finsler spaces of general dimension [5] and it was shown [7] that the second curvature tensor P_{ijkh} and other important tensors of a C-reducible Finsler space are of a special interesting form. But such an interesting form of the third curvature tensor R_{ijkh} is unknown yet.

The purpose of the present paper is to study C-reducible Finsler spaces under the assumption that R_{ijkh} satisfies some special conditions.

The notations and terminologies are used the ones of the monograph [4]. The original manuscript is prepared by C. Shibata with the advice and criticism of M. Matsumoto.

§ 1. Preliminaries

Let F^n be a Finsler space of dimension n with the fundamental function $L(x, y)$, $(y=\dot{x})$. We denote by $g_{ij}(x, y)$ the fundamental tensor $(\partial^2L^2/\partial y^i\partial y^j)/2$ and put $h_{ij}=g_{ij}-l_il_j$, where $l_i=\partial L/\partial y^i$.

DEFINITION. A Finsler space $F^n(n>2)$ is called *C-reducible*, if the torsion tensor C_{ijk} is of the form

$$C_{ijk}=(h_{ij}C_k+h_{jk}C_i+h_{ki}C_j)/(n+1), \tag{1.1}$$

where C_i is the torsion vector $C_{ijk}g^{jk}$.

It is obvious from (1.1) that $C_i=0$ is equivalent to $C_{ijk}=0$ (Riemannian) for a C-reducible Finsler space.

The v-curvature tensor S_{ijhk} of F^n is defined by

$$S_{ijhk}=C_{ikr}C_j{}^r{}_h-C_{ihr}C_j{}^r{}_k.$$

It is shown [5] that S_{ijhk} of a C-reducible Finsler space is of the form

$$S_{ijhk}=(h_{ik}C_{jh}+h_{jh}C_{ik}-h_{ih}C_{jk}-h_{jk}C_{ih})/(n+1)^2, \tag{1.2}$$

where we put $C_{ij}=(\beta/2)h_{ij}+C_iC_j$ and $\beta=g_{jk}C^jC^k$.

The h-curvature tensor $R_j{}^i{}_{hk}$ is defined by

$$\begin{aligned}R_j{}^i{}_{hk}=&(\partial_k\Gamma^*{}_j{}^i{}_h-G_k{}^r\dot{\partial}_r\Gamma^*{}_j{}^i{}_h)-(\partial_h\Gamma^*{}_j{}^i{}_k-G_h{}^r\dot{\partial}_r\Gamma^*{}_j{}^i{}_k)\\&+C_j{}^i{}_r(\partial_kG_h{}^r-\partial_hG_k{}^r+G_h{}^sG_s{}^r{}_k-G_k{}^sG_s{}^r{}_h)\\&+\Gamma^*{}_r{}^i{}_k\Gamma^*{}_j{}^r{}_h-\Gamma^*{}_r{}^i{}_h\Gamma^*{}_j{}^r{}_k,\end{aligned}$$

Received by the editors Mar. 15, 1976.

where $\Gamma^*{}_j{}^i{}_h$, $G_j{}^i{}_h$ and $G_k{}^r$ are well-known quantities. On the other hand, the curvature tensor $H_j{}^i{}_{hk}$ of Berwald is defined by

$$H_j{}^i{}_{hk}=\partial_k G_j{}^i{}_h-\partial_h G_j{}^i{}_k+G_j{}^r{}_h G_r{}^i{}_k-G_j{}^r{}_k G_r{}^i{}_h$$
$$+G_r{}^i{}_{jk}G_h{}^r-G_r{}^i{}_{jh}G_k{}^r,$$

where $G_r{}^i{}_{jk}=\dot{\partial}_k G_r{}^i{}_j$. These curvature tensors are in the well-known relation [8]

$$R_{ijkh}=(H_{ijkh}-H_{jikh})/2-g^{rs}(g_{ir(k)}g_{js(h)}-g_{ir(h)}g_{js(k)})/4,$$

where (k) denotes the covariant differentiation with respect to x^k in the sense of Berwald. In terms of the covariant differentiation $|k$ in the sense of Cartan and $|o=(|k)y^k$, it is written in the form

(1.3) $$R_{ijkh}=(H_{ijkh}-H_{jikh})/2-(C_{irk|o}C_j{}^r{}_{h|o}-C_{irh|o}C_j{}^r{}_{k|o}).$$

§ 2. The h-curvature tensor of a C-reducible Finsler space

A Finsler space is usually called a *Landsberg space*, if the condition $C_{ijk|o}=0$ holds good. Further, if the condition $C_{ijk|h}=0$ is satisfied, then the space is called a *Berwald space*. Theorem 1 of the paper [5] says that a C-reducible Landsberg space is a Berwald space.

LEMMA. *A C-reducible Finsler space is a Berwald space, if and only if the vector* $C_{i|o}$ *vanishes.*

Proof: Referring to the equation $h_{ij|k}=0$, we obtain from (1.1)

(2.1) $$C_{ijk|o}=(h_{ij}C_{k|o}+h_{jk}C_{i|o}+h_{ki}C_{j|o}).$$

Therefore we see that $C_{ijk|o}=0$ is equivalent to $C_{i|o}=0$. Consequently the proof of Lemma is complete.

Let us substitute from (2.1) into (1.3). Then the tensor R_{ijkh} is written in the form

(2.2) $$R_{ijkh}=(H_{ijkh}-H_{jikh})/2+h_{ik}H_{jh}+h_{jh}H_{ik}-h_{ih}H_{jk}-h_{jk}H_{ih},$$

where $H_{ik}=(C_{i|o}C_{k|o}+(\mu/2)h_{ik})/(n+1)^2$ and $\mu=C_{r|o}C^r{}_{|o}$.

THEOREM 1. *If the h-curvature tensor* R_{ijkh} *of a C-reducible Finsler space is equal to the Berwald curvature tensor* H_{ijkh}, *then the space is a Berwald space, under the assumption that the space be of dimension* $n\geqq 4$, *and that for* $n=3$ *the metric be positivedefinite.*

Proof: If $R_{ijkh}=H_{ijkh}$ holds good, it follows from (2.2) and $R_{ijkh}=-R_{jikh}$ that

(2.3) $$h_{ik}H_{jh}+h_{jh}H_{ik}-h_{ih}H_{jk}-h_{jk}H_{ih}=0.$$

We contract this by g^{ih} to obtain

(2.4) $$(3n-5)\mu h_{jk}+(n-3)C_{j|o}C_{k|o}=0,$$

and further contraction of (2.4) by g^{jk} leads us to $\mu=0$. Thus, if $n\geqq 4$, we have $C_{j|0}=0$ from (2.4). For $n=3$, we obtain the same result, if the quadratic form $\mu=g_{ij}C^{i}{}_{|0}C^{j}{}_{|0}$ is positive-definite. Consequently the space under consideration is a Berwald space by means of Lemma.

Now we consider two special cases. The one is "isotropy" of a Finsler space and the other is of R3-like. H. Akbar-Zadeh introduced the notion of isotropy to Finsler spaces [1] and showed that the h-curvature tensor R_{ijhk} of an isotropic Finsler space of dimension $n\geqq 3$ is written in the form

$$R_{ijkh}=R(g_{ik}g^{jk}-g_{ih}g_{jk}),\quad R=\text{const.} \tag{2.5}$$

Further he proved that if $R\neq 0$, then the v-curvature tensor S_{ijkh} vanishes. Therefore, if the distance function is symmetric, then the space is Riemannian by means of the well-known theorem due to F. Brickell. We shall show

THEOREM 2. *If a C-reducible Finsler space is isotropic at every* (x, y) *and the scalar* R *of* (2.5) *does not vanish, then the space is Riemannian under the assumption as in Theorem* 1.

Proof: Suppose that a C-reducible Finsler space be isotropic at every (x, y) and the scalar R do not vanish. It follows from $S_{ijkh}=0$ and (1.2) that

$$h_{ik}C_{jh}+h_{jh}C_{ik}-h_{ih}C_{jk}-h_{jk}C_{ih}=0. \tag{2.6}$$

The contraction of (2.6) by g^{jk} leads us to

$$(3n-5)\beta h_{ih}+(n-3)C_iC_h=0,$$

and once more contraction by g^{ih} gives $\beta=0$. Therefore we obtain $C_i=0$ (Riemannian) under the same assumption as in Theorem 1. Consequently the Theorem 2 is proved.

Next we are concerned with a C-reducible Finsler space with $R_{ijkh}=0$, which is the exceptional case in Theorem 2. We note that the tensor H_{ijkh} vanishes when the tensor R_{ijkh} does. Therefore, if $R_{ijkh}=0$ for a C-reducible Finsler space, it is seen from Theorem 1 that the space is a Berwald space. Consequently we obtain

THEOREM 3. *If the h-curvature tensor* R_{ijkh} *of a C-reducible Finsler space vanishes, then the space is (locally) Minkowski under the assumption as in Theorem* 1.

Finally one of the authors showed [6] that the h-curvature tensor R_{ijkh} is written in the form

$$R_{ijkh}=g_{ik}L_{jh}+g_{jh}L_{ik}-g_{ih}L_{jk}-g_{jk}L_{ih}, \tag{2.7}$$

if the dimension is equal to three. Thus a Finsler space of dimension n ($\geqq 4$) is called R3-like, if the equation (2.7) holds good. In the following we treat Finsler spaces with R_{ijkh} of the form (2.7). In virtue of (2.7), we have

$$\begin{aligned} H_{ijkh}=&(g_{ik}L_{jh}+g_{jh}L_{ik}-g_{ih}L_{jk}-g_{jk}L_{ih})\\ &-2C_i{}^r{}_j(y_kL_{rh}-y_hL_{rk})+y_k\dot\partial_jL_{ih}-y_h\dot\partial_jL_{ik}. \end{aligned} \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.2) we obtain

$$(2.9)\qquad h_{ik}H_{jh}+h_{jh}H_{ik}-h_{ih}H_{jk}-h_{jk}H_{ih}$$
$$+(K_{ihj}-K_{jhi})y_k-(K_{ikj}-K_{jki})y_h=0,$$

where we put

$$K_{ihj}=(2\dot{\partial}_jR_{ih}-g_{ih}\dot{\partial}_jR/(n-1))/(n-1),$$
$$R_{ik}=g^{jh}R_{ijkh}=(n-2)L_{ik}+Lg_{ik},$$
$$L=g^{ik}L_{ik},\quad R=g^{ik}R_{ik}.$$

If $K_{ihj}=0$, then (2.9) is reduced to (2.3), so that $C_{i|0}=0$ is derived. Therefore by Lemma and Theorem 1 we have

THEOREM 4. *If the tensor R_{ik} of a C-reducible Finsler space with R_{ijkh} of the form* (2.7) *satisfies the equation*

$$\dot{\partial}_jR_{ik}=g_{ik}\dot{\partial}_jR/2(n-1),$$

then the space is a Berwald space under the assumption as in Theorem 1.

It is remarked that L_{jh} in (2.7) is not necessarily symmetric and the so-called Ricci tensor R_{jh} is not necessarily so. The condition in Theorem 4 implies the symmetry property of $\dot{\partial}_jR_{ik}$ in i and k, so that $R_{ik}-R_{ki}$ are functions of positions only.

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