J. Korean Math. Soc.
Vol. 13, No. 1, 1976

THE EXACT EICHLER COHOMOLOGY SEQUENCE OF EXTENDED KLEINIAN GROUP

BY SANG MOON KIM

Introduction

There are many authors who have studied the cohomology of Kleinian groups introduced by Eichler [7], and they have obtained many important results which show the structure of discontinuous Kleinian groups.

In this paper we study the Eichler cohomology of extended groups. Following the method of Kra [10] and [11], we derive an exact cohomology sequence which coincides with the cohomology sequence for Kleinian groups obtained by Kra in the above papers.

In this paper E denotes a non-elementary extended group, and G is the maximal Kleinian group contained in E. E acts on the extended complex plane, let Λ be the set of it's limit points and let D be the complement of Λ then it is called as the *region of* discontinuity of E.

λ denotes the Poincare metric defined on each compnent of D with constant curvature -4.

If X is an open subset of the extended complex plane, and if f is ananalytic or anti-analytic ($\bar{f}$ is analytic), then for each function h on $f(x)$, define

$$(f^*_{r,s}h)(z)=h(f(z))f'(z)^r\overline{f'(z)}^s. \text{ for analytic } f,$$

$$(f^*_{r,s}h)(z)=\overline{h(f(z))f'(z)}^r f'(z)^s \text{ for antianalytic } f.$$

In the above f' denotes $\frac{\partial f}{\partial z}+\frac{\partial f}{\partial \bar{z}}$. We abbreviate $f_{r,o}{}^*$ by $f_r{}^*$.

A *measurable automorphic form* of weight$(-2q)$ is a measurable function on D that satisfies $A_q{}^*f=f$ for every A in E.

For each p with $1\leq p\leq\infty$, the measurable automorphic forms with

$$\|f\|_{q,\infty}=\sup_{z\in D}\lambda(z)^{-q}|f(z)|<\infty \text{ for } p=\infty$$

(or $$\|f\|^p_{p,q,D/E}=\iint_{D/E}\lambda(z)^{2-qp}|f(z)|^p|dz\wedge d\bar{z}|<\infty$$

for $p\neq\infty$) form a Banach space $L_q{}^\infty(D,E)$ (or $L_q{}^p(D,E)$) of *bounded forms* (or *p-integrable forms*). A *holomorphic form* is a holomorphic measurable automorphic form which satisfies $\lim_{z\to\infty} f(z)=0(|z|^{-2q})$ if $\infty\in D$.

For $p\geq 1$, Ithe holomorphic forms in $L_q{}^p(D,E)$ form a closed subspace, denoted by $A_q{}^p(D,E)$.

If $g\in L_q{}^\infty(D,E)$ then $\lambda^{2-2}{}_q\bar{g}$ is calledas a generalized *Beltrami coefficient*. A continuous function F on complexplane will be called a *portential* for the Beltrami coefficient if it

Received by the editors Feb. 28, 1976.

satisfies

$$\frac{\partial F}{\partial \bar{z}}=\lambda^{2-2}{}_q\bar{g} \text{ and } F(z)=0\ (|z|^{2q-2}),\ z\to\infty$$

Let Π_{2q-2} be the vector space of complex polynomials degree at most $2q-2$.

Put $PA=A^*{}_{1-q}P$ for all$P\in\Pi_{2q-2}$ and $A\in E$. This defines an action of the group on the vector space Π_{2q-2}. A mapping

$$\chi: E\to\Pi_{2q-2} \text{ is a } \textit{cocycle} \text{ if}$$

$$\chi(AB)=\chi(A)B+\chi(A), \text{ for all } A \text{ and } B \text{ in } E.$$

If $p\in\Pi_{2q-2}$, its *coboundary* is the cocycle

$$\chi(A)=PA-P,\ A\in E.$$

The space of cocycles factored by the space of coboundaries is called as the first *cohomology group* and denote it by $H^1(E,\ \Pi_{2q-2})$.

If the Riemann surface D/G contains an open set N homeomorphic to $\Delta=\{z:\ 0<|z|<1\}$, and if there is no point x on D/G which may corresponds to zero of the complex unit disk in extending the above homeomorphism, then the hole of D/G is called a puncture. If the puncture comes from a fixed point of parabolic transformations, then we call it a *parabolic puncture*.

Let A be a parabolic element in E, then considering the conjugate group of E, we may consider A as a generator of all parabolic elements fixing ∞, further we may assume $A(z)=z+1$. Then we can choose a fundamental domain F, so that F contains

$$V=\{z: |z|>\alpha,\ 0\geq \mathrm{Im}(z)>1, \text{ for some } \alpha>0\}.$$

The above V is called as a *cusped region*.

Let M be a subset of D, for two functions g and h defined on M, set

$$(k, g)_{q,M}=\iint_M \lambda(t)^{2-2q}k(t)g(t)\ dt\wedge d\bar{t},$$

if M coincides with D, then delete the symbol D in the above. For other undefined terms refer Kra [12], Kim [8] and Kim [9].

2. Results.

Now on, D denotes an invariant (equivalent to $A(D)=D$ for all A in E) union of open subsets of the region of discontinuity of E. Let G be the maximal Kleinian group contained in E. We need the following lemma proved by Kra in [11]

LEMMA 1. *There is a smooth function η on D which satisfies the following conditions*:

(1) $0\leq\eta\leq 1$,

(2) *for each $z\in D$, there is a neighborhood U of z and a finite subset J of G such that $\eta B(U)=0$ for each $B\in G-J$,*

(3) $\sum_{B\in G}\eta(B(z))=1,\ z\in D,$ and

(4) *If Y is a fundamental domain for G in D, then for each puncture p on D/G, we may choose a cusped region $V\subset Y$ belonging to p, such that $\eta B(V)=0$ for each $B\in G-\{1,A\}$, where A is the parabolic transformation corresponding to p.*

Consider the cohomology group of E, E acting on the vector space of complex polynomials degree at most $2q-2$.

Let p be a cocycle that represents a cohomology class of $H^1(E,\Pi_{2q-2})$, $q\geq 2$. Fix a fundamental domain $X\cup U(X)$ of G where X is a fundamental domain of E and $U\in E$ is an antianalytic element. Let K be a smooth function which satisfies the following conditions:

(5) $p(A)=A^{*}{}_{1-q}K-K$ on D, $A\in E$,

(6) If t is a fixed point of a parabolic element of E which corresponds to a puncture on D/E and t is contained in the closure of $X\cup U(X)$ then

$$K(z)=0(|z|^{2q-2+\varepsilon}),\quad z\to t\ (\text{if } t=\infty),$$

and

$$K(z)=0(|z-t|^{-\varepsilon}),\quad z\to t\ (\text{if } t\neq\infty),$$

in a cusped region (contained in $X\cup U(X)$) belonging to t,

(7) $k(z)=\partial K/\partial \bar{z}$, $z\in D$, then k is a generalized Beltrami coefficient for E.

If $\varepsilon=1$, we shall say that K *represents p quasi-boundedly on D*; and if $\varepsilon=0$, we shall say that K *represents p boundedly*on D.

A cohomology class of $H^1(E,\Pi_{2q-2})$ is called *parabolic* if for everry parabolic transformation $A\in G$ coresponding to a puncture on D/G, there is a polynomial $P\in\Pi_{2q-2}$ such that

$$\chi(A)=PA-P \text{ for all } A\in E,$$

for a cocycle χ that represents the cohomology class. The space of all parabolic cohomology classes is denoted $PH^1(E,\ \Pi_{2q-2})$.

LEMMA 2. *Let E be an extended group such that D/E is of finite type. If p is a cocycle tha trepresents a cohomology class of $H^1(E,\Pi_{2q-2})$, $q\leqq 2$, then there exsists a smooth function K on D that represents p quasi-boundedly on D. If p represents a cohomology class of $pH_D{}^1(E,\Pi_{2q-2})$ then K may be chosen to represent p boundedly on D.*

Proof. Let η be a function which satisfies all of the conditions of the Lemma 1, for G. Let G be the maximal Kleinian group contained in E. Then $\frac{1}{2}\eta$ is a smooth function on D and it satisfies (1), (2) and (3) for E. Set

$$K(z)=-\sum_{A\in E}\frac{1}{2}\eta(A(z))(P(A))(z),\quad z\in D. \tag{8}$$

Then by a simple calculation we have

$$(9) \qquad (A^*_{1-q}K)\ (z)-K(z)=(P(A))(z) \text{ for all } A\in E.$$

By Kra [12], we can choose $X\cup U(X)$ to be a finite union of relatively compact subsets of D and a finite number of cusped regions belonging to the punctures on D/E. If D/E is compact then we can choose $X\cup U(X)$ compact, hence K defined by (8) represents p boundedly on D. If the group E is finitely generated, then it is known that the surface D/E has only a finite number of punctures. Assume D/E has only one puncture. It is sufficient to assume ∞ is the fixed point corresponding to the puncture in $X\cup U(X)$; the corresponding parabolic element is $A(z)=z+1$,

$$(10) \qquad V_e=\{z\in C: 0\leq \text{ Real } z<1,\ \text{Im}(z)>e\}.$$

By Kra [11], there is a polynomial V of degree $\leq 2q-1$, such that

$$(p(A))(z)=V(z+1)-V(z),\ \ z\in C.$$

Let $f(A)=p(A)-(A^*_{1-q}\ V-V)$, $A\in E$, and $H(z)=-\sum_{A\in G}\frac{1}{2}\eta(A(z))\ \ (f(A))(z)$. Then by Kra [11], we have

$$H(z)\,|\,U_e\equiv 0.$$

$$\text{Let } K(z)=-\sum_{A\in E}\frac{1}{2}\eta(A(z))\ \ (f(A))(z)$$

$$=-\sum_{A\in G}\eta(A(z))(f(A))(z)-\sum_{A\in GB}\frac{1}{2}\eta(A(z))(f(A))(z)$$

$$=H(z)+H(B(z))B'(z)^{1-q}-\frac{1}{2}(f(B(z)).$$

where B is anarbitrary anti-analytic element in E. If ∞ corresponds to a puncture contained in the boundary of D/E then by (2.6) there is an anti-analytic element B in E which maps U_e into U_e. For the B we have

$$H(B)(z))\,|\,U_e\equiv 0 \text{ and } K(z)\,|\,U_e=-\frac{1}{2}(f(B))(z).$$

We proved that $K(z)\,|\,U_e$ is a polynomial of degree at most $2q-2$.

Now if ∞ corresponds to a puncture which is not contained in the boundary of D/E then we do not know the value $H(B(z))$, hence we use the other construction. Let η be a smooth function which satisfies (1), (2) and (3) for the group E. We further require that satisfy (4) for the puncture corresponding to ∞ and for the group E. η can be constructed by repeating the process of constructing such a function for a Kleinian group. Now repeat the construction of $H(z)$with E and instead of G and $\frac{1}{2}\eta$, and denote this function by $K(z)$. Then we have

$$K(z)\,|\,U_e \equiv 0,$$

and $K+v$ represents p quasi-boundedly on D in both cases. If p is parabolic then we can choose v in Π_{2p-2}; and $K+v$ represents p boundedly on D. If there is more than one puncture, then we can construct K by induction, following Kra [11].

We shall say that the surface D/G (or D/E) is of *finite type* if (a) D/G consists of finitely may disconnected surfaces, (b) the projection $D \rightarrow D/G$ is ramified over a finite number of points and (c) for each connected component U of D/G, there exist a compact surface S such that $S-U$ consists of finite number of points.

It is known that E is finitely generated then D/G and hence D/E is of finite type.

The following lemma is proved by Kra for Kleinian groups, but it naturally extends to extended Kleinian groups.

LEMMA 3. *Let E be a (non-elementary) extended group and D be an invariant union of components of its region of discontinuity such that D/E is of finite type. Then there is a canonical real-linear mapping*

$$\beta: H^1(E, \Pi_{2q-2}) \rightarrow A_q^{\infty}(D, E), \quad q \geq 2.$$

Proof. The proof is a natural exension of the proof of Kleinian case, hence we show only how *Kra's map* β is constructed.

Let p be a cocycle representing a cohomology class in $H^1(E, \Pi_{2q-2})$. Choose a smooth function K which represents p quasi-boundedly on D.
Let

$$(k(z) = \frac{\partial K}{\partial \bar{z}}, \quad z \in D$$

and

$$l(g) = \iint_{D/G} g(z) k(z) dz \wedge d\bar{z}, \quad g \in A_q^1(D, E).$$

Then l ls a bounded real linear imaginary valued functional on $A_q^1(D, E)$; by [8], we identify l with an element h of $A_q^{\infty}(D, E)$, define

$$\beta(p) = h.$$

A holomorphic function F on D is called a *holomorphic Eichler integral of order* $(1-q)$ on D for the group E if

(11) $$p(A) = A^*{}_{1-q}F - F \in \Pi_{2q-2}, \text{ all } A \in E$$

where q is an integer ≥ 2. An Echler integral F is called *bounded* if

$$f = D^{2q-1}F \in A_q^{\infty}(D, G).$$

The space of bounded Eichler integrals (modulo Π_{2q-2}) is denoted by $E^b{}_{1-q}(D, E)$. An Eichler integral F is called *quasi-bounded* if $f = D^{2q-1}F$ projects to a meromorphic q-differential Φ that can be extended to D/E with

$$\text{order}_p \Phi \geq -q.$$

The space of quasi-bounded Eichler integrals (modulo Π_{2q-2}) is denoted by $E^c{}_{1-q}(D, E)$.

The following theorem is also a natural extension of the corresponding theorem of the Kleinian case.

THEOREM 4.4. *Let E be a (non-elementary) extended group and D be an invariant union of components of its region of discontinuity such that D/E is of finite type. Then there exists an injective real linear mapping*

$$\alpha: E^c{}_{1-q}(D, E) \to H^1(E, \Pi_{2q-2})$$

such that

$\alpha(f) \in pH_D{}^1(E, \Pi_{2q-2})$ if and only if $f \in E^b{}_{1-q}(D, E)$.

We describe only how the map α is defined. Let $f \in E^b{}_{1-q}(D, E)$ and let F be a representative of f, then (11) defines a cocycle $\alpha(f)$ whose cohomology class depends only on f.

THEOREM 4.5. *Let D, E and D/E be as in the above theorem. Then the following is a commutative diagram with exact rows;*

(12)
$$\begin{array}{ccccccccc} 0 \longrightarrow & E^b{}_{1-q}(D, E) & \xrightarrow{\alpha} & pH^1{}_D(E, \Pi_{2q-2}) & \xrightarrow{\beta} & A_q{}^\infty(D, E) & \xrightarrow{0} \\ & \downarrow i_1 & & \downarrow i_2 & & \downarrow i_3 & \\ 0 \longrightarrow & E^c{}_{1-q}(D, E) & \xrightarrow{\alpha} & pH^1(E, \Pi_{2q-2}) & \xrightarrow{\beta} & A_q{}^\infty(D, E) & \longrightarrow 0 \end{array}$$

where i_1, i_2 and i_3 are inclusion maps.

Proof. Everything is proved except that image

$$\alpha = \text{kernel } \beta$$

It is clear that image α is contained in the kernel of β.

Let p be a cocycle that represents a (parabolic) cohomology class of $H^1(E, \Pi_{2q-2})$. Choose a smooth function K that represents p (boundedly) quasi-boundedly on D. Let

$$k(z) = \frac{\partial K}{\partial \bar{z}}, \quad z \in D.$$

Suppose $\beta(p) = 0$; that is

(13) $$l(g) = \iint_{D/G} g(z) k(z) dz \wedge d\bar{z} = 0 \text{ for all } g \in A^1{}_q(D, E).$$

Let F be the potential of k defined by

$$F(z) = \frac{(z-a_1)\cdots(z-a_{2q-1})}{2\Pi i} \iint_D \frac{\lambda(t)^{2-2q} \bar{k}(t) dt \wedge d\bar{t}}{(t-z)(t-a_1)\cdots(t-a_{2q-1})},$$

where $\{a_1, a_2, \cdots a_{2q-1}\}$ are $2q-1$ different points contained in the limit set of E. Let

$$h^z(t) = -\frac{1}{2it} \frac{(z-a_1)(z-a_2)\cdots(z-a_{2q}-1)}{(t-z)(t-a_1)\cdots(t-a_{2q-1}),} t \in D.$$

Then we havethe following identity, see Kim [11], (or one can prove ti by simple calculation),

$$F(z) - \overline{F(z)} = (h^z, k)_q - \overline{(h^z, k)}_q = (\Theta_q h^z, k)_q, D/G.$$

Here $\theta_q h^z = \sum_{A \in E} A^*{}_q h^z$ (Poincarē operator).

Since $\Theta_q h^z \in A_q{}^1(D, E)$, by (12) and by the above identity we have

$$F(z) - \overline{F(z)} = 0 \text{ for all } z \in \Lambda - \{a_1, \cdots a_{2q-1}\}.$$

That is $F(z)$ is real for all $z \in \Lambda - \{a_1, \cdots a_{2q-1}\}$.
Let $c = x + iy$ then

$$cF - \bar{c}\overline{F(z)} = (\Theta_q c h^z, k)_{q, D/G} = 0.$$

Hence $yF(z) = -yF(z)$ and we have $F(z) = 0$ on Λ. We conclude that

$$\partial F / \partial \bar{z} = k \text{ and } A^*{}_{1-q}F - F = 0.$$

Then $K - F$ is holomorphic on D, $K - F \in E^c{}_{1-q}(D, E)$, and $K - F \in E^b{}_{1-q}(D, E)$ if p is parabolic. By construction, we have $\alpha(K-F) = p$ and it shows that kernel of β is contained in the image of α.

References

[1] L. V. Ahlfors, "*Finitely generated Kleinian groups*", Amer. J. Math., **86**(1964), and **87** (1965).

[2] L. V. Ahlfors, "*Eichler integrals and Bers' area theorem*", Mich. Math. J., **15**(1968).

[3] L. Bers, "*Automorphic forms and Poincaré series for infinitely generated Fuchsian groups*", Amer. J. Math., **87**(1965).

[4] L. Bers, "*An approximation theorem*", J. Analyse Math., **14**(1965).

[5] L. Bers, "*Inequalities for finitely generated Kleinian groups*", J. Analyse Math., **18**(1967).

[6] L. Bers, "*On Ahlfors' finiteness theorem*", Amer. J. Math., **89**(1967).

[7] M. Eichler, "*Eine Verallgemeinerung der Abelschen Integrale*", Math. Z., **67**(1957).

[8[S. M. Kim, "*Bers' duality of automorphic forms of extended Kleinian group*", Kyungpook Math. J., **14**(1974)

[9] S. M. Kim, "*A note on the Eichler cohomology of extended Kleinian group*" (to appear).

[10] I. Kra, "*On cohomology of Kleinian groups*", Ann. of Math., **89**(1969).

[11] I. Kra, "*On cohomology of Kleinain groups: II*", Ann. of Math. **90**(1969).

[12] I. Kra, "*Automorphic forms and Kleinian groups*", W. A. Benjamin, NewYork, (1971).

[13] J. Lehner, "*A shortcourse in automorphic functions*", Holt, Reinhart and Winston, New York, (1965).

Seoul National University