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A NOTE ON THE EICHLER COHOMOLOGY OF EXTENDED KLEINIAN GROUP

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Introduction

An extended Kleinian group E acts on the vector space of complex polynomials. We can define the first cohomology space of the group with coefficient in the vector space. Kra [6] and [7], studied the cohomology in Kleinian case. Since the extended group E contains a maximal Kleinian subgroup G, the two cohomology have close relations. We prove that the two cohomology have same dimmension and a portential function for the generalized Beltrami coefficient related to E satisfies an identity which can be described in terms of the Kleinian group G.

Ane xtended Kleinian group E is a group of Möbius transformations and antianalytic transformations $\frac{a\bar{z}+b}{c\bar{z}+d}$ which acts discontinuously on an open subset of the complex plane. Let G be the set of all *M*öbius transformations contained in E, then the Kleinian group G is a normal subgroup of the extended group E. The subgroup G has index two in E.

E acts on the extended complex plane $C\cup\{\infty\}$. If there are distinct elements $\{A_n\}$, $A_n\in E$, and a point $p\in C\cup\{\infty\}$ such that $\lim_{n\to\infty} A_n(z)=p$ then we say p is a *limit point* of the group E. In the above, if $A_n\in G$ for all n then we call p a *limit point* of G.

If U and V are two anti-analytic elements of E then $UV\in G$, hence if p is a limit point of E, then it is also a limit point of G.

Let D be the complement of the set of limit points of E; we call D the *region of discontinuity* of E.

If the region of discontinuity is not empty then we call E an *extended Kleinian group*, (in the future this will be called an *extended group*).

Let f be a complex valued unction defined on an open set in the complex plane. If the conjugate function$\bar{f}$ is analytic then we call f an *anti-analytic function.*

The orbit space D/E forms (not necessarily connected)a surface with boundary and it is known that almost all surfaces arise in this way. D/E carries a well known complex structure in which coordinate transformations are analytic or anti-analytic. D/E with this complex structure is called a *Klein surface.*

Let f be an assignment of a meromorphic function f_α on each coordinate neighborhood (U_α, Z_α) of D/E, where Z_α isa local uniformizer on U_α.

If, for every pair of coordinate neighborhoods U_α and U_β with non-empty interesection, f satisfies

$$f_\alpha(dZ_\alpha)^q=f_\beta(dZ_\beta)^q$$

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whenever the uniformizers Z_α and Z_β are analytically related, and

$$f_\alpha(dZ_\alpha)^q=\overline{f_\beta(dZ_\beta)^q}$$

whenever Z_α and Z_β are anti-analytically related, we call f a *meromorphic differential of order q*. If $q=0$ we call f a *meromorphic function*.

The meromorphic differentials of order q lifted to D are called as automorphic forms of weight $-2q$.

Let D be the region of discontinuity of E. If D is not empty then we call E *discontinous*, and if D does not contain at least three points of $C\cup\{\infty\}$ then we call E a *(non-elementary)* extended group.

In this paper we study only (non-elementary) extended group.

Let X be a fundamental domainof an extended group $E=G\cup GU$. Then it is clear that $X\cup U(X)$ forms a fundamental domain of G. (For the definition of fundamental domain see [8]).

Let E act on the upper half complex plane. By the same technique as for Fuschian groups, we can choose a fundamental domain of E: a polygon bounded by analytic Jordan arcs. By well known methods (e.g. [8]), we can determine the fundamental domain of a general (non-elementary) extended group, specifically, it is the countable union of disjoint polygons bounded by analytic Jordanarcs, and we can take one polygon from each inequivalent connected component of the region of discontinuity.

Let D be an open set in the extended complex plane and f be an analytic or anti-analytic mapping of D into the extended complex plane. Let r and s be integers. Then for every function h on $f(D)$, we define a function $f^*_{r,s}h$ on D by

$$(f^*_{r,s}h)(z)=h(f(z))f'(z)^r\overline{f'(z)}^s$$

for analytic f, and

$$(f^*_{r,s}h)(z)=\overline{h(f(z))f'(z)^r}f'(z)^s$$

where f' denotes $\frac{\partial f}{\partial \bar{z}}$, for anti-analytic f. We abbreviate $f^*_{r,o}$ by f^*_r. It is clear that: $(f\circ g)^*_{r,s}=g^*_{r,s}\circ f^*_{r,s}$. Let E be a (non-elementary) extended group and D be an invariant union of components of the region of discontinuity of E. Let λ_D be the *Poincaré metric* (the unique complete conformal Riemannian metric defined on each component of D with constant curvature-4). For any conformal or anti-conformal map f of D, we have $\lambda_{f(D)}(f(z))\ |f'(z)|=\lambda_D(z)$, all $z\in D$.

Let $q\geq 2$ be a fixed integer. A *measurable automorphic form of weight* $(-2q)$ is a measurable function f that satisfies $A^*_q f=f$, for all $A\in E$. For each p with $1\leq p\leq\infty$, the measurable automorphic forms with

$$\|f\|_{q,\infty}=\sup_{z\in D}\lambda(z)^{-q}|f(z)|<\infty \quad \text{for } p=\infty$$

(or $$\|f\|^p_{q,pD/E}=\iint_{D/E}\lambda(z)^{2-qp}|f(z)|^p|dz\wedge d\bar{z}|<\infty \quad \text{for } p\neq\infty)$$

form a Banach space $L^\infty_q(D,E)$ (or $L^p_q(D,E)$) of *bounded forms* (or *p-integrable forms*). A *holomorphic form* is a holomorphic measurable automorphic form which satisfies $\lim_{z\to\infty} f(z)=0(|z|^{-2q})$ if $\infty\in D$. For $p\geq 1$, the holomorphic forms in $L_q{}^p(D,E)$ form a closed subspace, denoted by $A^p{}_q(D,E)$.

We use $\wedge$ to denote the set of all limit points of E.

2. Results

Let Π_{2q-2} denote the vector space of complex polynomials in one variable of degree at most $2q-2$. The group E acts on the right on Π_{2q-2} via

$$PA=A^*{}_{1-q}P,\quad P\in\Pi_{2q-2},\quad A\in E.$$

To verify that the above equation defines an action of E on Π_{2q-2}, let $A(z)=\dfrac{az+b}{cz+d}$ (or $\dfrac{a\bar{z}+b}{c\bar{z}+d}$) with $ad-bc=1$, and let $p(z)=z^n$. Then

$$(PA)(z)=(az+b)^n(cz+d)^{-n+2q-2}$$

(or $(PA)(z)=(\bar{a}z+\bar{b})^n(\bar{c}z+\bar{d})^{-n+2q-2}$.

Thus $PA\in\Pi_{2q-2}$ whenever $n\leq 2q-2$, hence whenever $P\in\Pi_{2q-2}$.

A mapping $\chi:E\to\Pi_{2q-2}$ is called a *cocycle* if $\chi(AB)=\chi(A)B+\chi(A)$, A and B in E.

If $p\in\Pi_{2q-2}$, its *coboundary*is the cocycle $x(A)=PA-P$, $A\in E$.

The *first cohomology group* $H^1(E,\Pi_{2q-2})$ is the space of cocycles factored by the space of coboundaries.

A relation between cohomology groups $H^1(E,\ \Pi_{2q-2})$ and $H^1(G,\Pi_{2q-2})$ is obtained by the well known homological-algebra argument for $q\leq 2$, where E is a (non-elementary) extended group and G is the maximal Kleinian group in E.

THEOREM 1. *Let E be a non-elementary extended group and G is the maximal Kleinian group in E, then we have the following identity;*
Real dim $H^1(E,\Pi_{2q-2})$=Complex dim $H^1(G,\Pi_{2q-2})$.

Proof: From a well known spectral sequence of cohomology;

(1)
$$0\to H^1(E/G,\Pi^G{}_{2q-2})\to H^1(E,\Pi_{2q-2})$$
$$\to H^1(G,\Pi_{2q-2})^{E/G}\to H^1(E/G,\ \Pi^G{}_{2q-2})\to$$

we have

(2)
$$H^1(E,\Pi_{2q-2})\cong H^1(G,\Pi_{2q-2})^{E/G}$$

because $\Pi^G{}_{2q-2}=0$, so that the 1st and the 4th term inthe sequence vanishes.

In the above, the operation of E/G on $H^1(G,\ \Pi_{2q-2})$is induced by:

(3)
$$u(f)(B)=f(UBU^{-1})U$$

where u is the generator of E/G, U is a fixed element in E-G and f is a cocycle of $H^1(G,\Pi)$.

Since E/G has order two, $H^1(G, \Pi_{2q-2})$ splits into the sum of the $(+1)$-eigenspace $H^1(G,\Pi_{2q-2})_+=H^1(G,\Pi_{2q-2})^{E/G}$ of u, and the (-1)-eigenspace $H^1(G,\Pi_{2q-2})_-$ of u. Namely

$$H^1(G,\Pi_{2q-2})=H^1(G,\Pi_{2q-2})_+\oplus H^1(G,\Pi_{2q-2})_-.$$

Because of (3), it is clear that

$$u(af)=\bar{a}u(f) \text{ for any complex number } a. \tag{4}$$

Let f be a cocycle of $H^1(G,\Pi_{2q-2})_+$ then if is a cocycle of $H^1(G,\Pi_{2q-2})_-$ and vice versa. Hence

Real dim $H^1(G,\Pi_{2q-2})_+=$Real dim $H^1(G,\Pi_{2q-2})_-$, therefore

Real dim $H^1(E,\Pi_{2q-2})=\frac{1}{2}$Real dim $H^1(G,\Pi_{2q-2})=$Realdim $H^1(G,\Pi_{2q-2})^{E/G}=$Complex dim $H^1(G,\Pi_{2q-2})$.

Let D be an invariant union of components of the region of discontinuity of E. By a *generalized Beltrami coefficient*, we mean a measurable function f on D such that

$$A^*_{1-q,1}f=f \text{ all } A\in E$$

and

$$|f|\leq \text{constant}\lambda^{2-q}, \text{ almost everywhere.}$$

If $g\in L_q^\infty(D,E)$ then $\lambda^{2-2q}\bar{g}$ is a generalized Beltrami coefficient. We consider a generalized Beltrami coefficient f to be defined on the entire plane and to vanish off D. A continuous function F on C will be called a *potential* for the generalized Beltrami coefficient $\lambda^{2-2q}\bar{g}$, if

$$F(z)=0(|z|^{2q-2}),\quad z\to\infty$$

and

$$\frac{\partial F}{\partial\bar{z}}=\lambda^{2-2q}\bar{g}.$$

For the construction of portential, we quote the following lemma from Kra [11].

LEMMA 1. *Let* $q\geq 2$ *and* $g\in L^\infty{}_q(D,E)$. *If* $\{a_1,a_2,\cdots a_{2q-1}\}$ *are distinct points in the limitset of E, then*

$$F(z)=\frac{(z-a_1)\cdots(z-a_{2q-1})}{2\pi i}\iint_D\frac{\lambda(t)^{2-2q}\bar{g}(t)dt\wedge d\bar{t}}{(t-z)(t-a_1)\cdots(t-a_{2q-1})} \tag{5}$$

$z\in C$ *is a potential for* $\lambda^{2-2q}\bar{g}$.

Note that $\frac{\partial F}{\partial\bar{z}}=\lambda^{2-2q}\bar{g}$ in the sense of generalized derivatives.

Let h be a holomorphic function on D, then we can define the *Poincaré series* of h

by

(6) $$(\theta_q h)(z)=\sum_{A\in E}(A^*{}_q h)(z)^q, \quad z\in D.$$

Whenever the right side converges uniformly and absolutely on compact subsets of D. It is well known that θ_q is a continuous real linear mapping of $A_q{}^1(D)$onto $A_q{}^1(D,E)$, for the above consult Kim [5], or Kra [8].

Let M be a subset of D, for two function g and k defined on M, we set

(7) $$(k,g)_{q,N}=\iint_M \lambda(t)^{2-2q} k(t)\overline{g(t)}\, dt\wedge d\bar{t},$$

if M coincide with D, then we delete the sign in (7).

Let $g\in L_q{}^\infty(D,E)$ and let F be the potential for $\lambda^{2-2q}\bar{g}=f$, defined by (5); let

(8) $$h^z(t)=\frac{1}{2\pi i}\cdot\frac{(z-a_1)\cdots(z-a_{2q-1})}{(t-z)(t-a\ )\cdots(-a_{2q-1})}, \quad t\in D.$$

Then for $z\in\Lambda-\{a_1,\cdots,a_{2q-1}\}$, where the $a_1,\ ,a_{2q-1}$ are distinct, $h^z(t)\in A_{q1}(D)$ and hence $\theta_q h^z\in A_q{}^1(D,E)$.

We will prove an identity that the difference of portential $F-\bar{F}$ can be represented in terms of $\theta_q h^z$, f and a fundamental domain of the group G. It means that as a portential of $\lambda^{2q-2}\bar{g}$, F is determined by quantities related to E, but $F-\bar{F}$ can be represented in terms of G only.

THEOREM 2. *Let* $g\in L^\infty{}_q(D,E)$, *then*

$$F(z)-\overline{F(z)}=(h^z,g)_q-(\overline{h^z,g})_q=(\theta_q h^z,g)_{q,XUB(X)}$$

where B *is an anti-analytic element in* E *and* X *isa a fundamental domain for* E, F *defined by* (5) *and* h^z *by* (8).

Proof. The identity is proved by some what involved calculations:

(9) $$\begin{aligned}(\theta_q h^z,f)_{p,X}&=\sum_{A\in G}\iint_X \lambda(t)^{2-2q}h^z(A(t))A'(t)^q\overline{g(t)}dt\wedge d\bar{t}\\&+\sum_{A\in GB}\iint_X \lambda(t)^{2-2q}\,\overline{h^z(A(t))A'(t)^q g(t)}dt\wedge d\bar{t}\\&=\sum_{A\in G}\iint_{A(X)}\lambda(t)^{2-2q}h^z(t)\overline{g(t)}dt\wedge d\bar{t}\\&+\sum_{A\in GB}\iint_{A(X)}\lambda(t)^{2-2q}\overline{h^z(t)}g(t)dt\wedge d\bar{t}.\end{aligned}$$

Since $\theta_q h^z\in A_q{}^1(D,E)$ we have
$(\theta_q h^z,g)_{q,X}=-(\theta_q h^z,g)_{q,B(X)}$ by simple calculation.
Hence

(10) $$(\theta_q h^z,g)_{q,X\cup B(X)}$$

$$= (\theta_q h^z, g)_{q,X} - \overline{(\theta_q h^z, g)}_{q,X}$$

By (9) and (10) we have

$$(\theta_q h^z, f)_{q, X \cup U(X)} = \iint_D \lambda(t)^{2-2q} h^z(t) \overline{g(t)} dt \wedge d\bar{t}$$

$$+ \iint_D \lambda(t)^{2-2q} \overline{h^z(t)} g(t) dt \wedge d\bar{t}$$

$$= (h^z, g)_q - \overline{(h^z, g)}_q = F(z) - \overline{F(z)}.$$

Let $\overline{D/G}$ be the compactification of the Riemann surface D/G, if there is a point x on $\overline{D/G}$ which has a neighborhood N such that $N \cap D/G$ is homeomorphic to $\Delta = \{z : 0 < |z| < 1\}$,
then we call as *puncture* on D/G.

A cohomology class

$Q \in H^1(E, \Pi_{2q-2})$ is called *D-parabolic* if for every parabolic transformation $A \in G$ corresponding to a puncture on D/G, there is a $p \in \Pi_{2q-2}$ such that

$$\chi(A) = PA - P$$

for some cocycle that represents Q. The space of D-parabolic cohomology classes is denoted by $pH_D{}^1(E, \Pi_{2q-2})$. Let $g \in L_q{}^\infty(D, E)$ and $f = \lambda^{2-2q}\bar{g}$. Let F be the potential for f, set

$$\chi(A) = A^*{}_{1-q}F - F, \text{ for all } A \in E \tag{11}$$

then χ is a cocycle. The cohomology class defined by (11) depends only on g, and it is denoted by $\beta^*(g)$. We call β^* *Bers' map.*

THEOREM 3. *There is a canonical real linear mapping*

$$\beta^* : L_q{}^\infty(D, E) \to pH_D{}^1(E, \Pi_{2q-2}).$$

Proof. β^* is defined by (11) and one needs to prove only that the $\chi(A)$ is parabolic, but the proof can be carried out exactly the same way as in the Kleinian case, see Kra [8].

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