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SUBMANIFOLDS AND THE LENGTH OF THE SECOND FUNDAMENTAL TENSORS

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§1. Introduction. In the previous paper [2], the author proved the following

THEOREM A. *Let M be a complete, connected submanifold of dimension $n(\geqq 3)$ immersed in an $(n+p)$-dimensional Riemannian manifold of positive constant curvature whose mean curvature vector field is parallel with respect to the induced connection of the normal bundle. If the second fundamental tensors H_A satisfy*

$$\sum_{A=1}^{p} \operatorname{trace} H_A{}^2 < \frac{1}{n-1} \sum_{A=1}^{p} (\operatorname{trace} H_A)^2, \tag{1.1}$$

then, M is umbilical with respect to the mean curvature normal direction. Furthermore, if the ambient manifold is an $(n+p)$-dimensional sphere, M is a minimal submanifold of a small sphere.

The purpose of the present paper is to prove a stronger theorem, that is, to prove the following

THEOREM. *Let M be an n-dimensional $(n\geqq 3)$, complete, connected submanifold of an $(n+p)$-dimensional Riemannian manifold of positive constant curvature c whose mean curvature vector field is parallel with respect to the induced connection of the normal bundle. If M is immersed without minimal point and for a real number δ, $0<\delta<c$, the second fundamental tensors H_A $(A=1,2,\cdots,p)$ satisfy*

$$\sum_{A=1}^{p} \operatorname{trace} H_A{}^2 < \frac{1}{n-1} \sum_{A=1}^{p} (\operatorname{trace} H_A)^2 + 2\delta, \tag{1.2}$$

then M is umbilical with respect to the mean curvature normal direction.

§2. Submanifolds of a space of constant curvature.

Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $\bar{M}$ of constant curvature $c>0$. The Riemannian connections of M and $\bar{M}$ are denoted by ∇ and $\bar{\nabla}$ respectively whereas the connection of the normal bundle of M in $\bar{M}$ is denoted by D. Let $N_1, \cdots N_p$ be mutually orthogonal unit normal vectors at a point p of M and extend them to local fields in a neighborhood of p. Then we have the following equations of Gauss and Weingarten:

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{A=1}^{p} g(H_A X, Y) N_A, \tag{2.1}$$

$$\bar{\nabla}_X N_A = -H_A X + D_X N_A, \tag{2.2}$$

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where X, Y are tangent vectors at p, g the Riemannian metric of M induced from that of $\bar{M}$ and H_A the second fundamental tensor with respect to N_A. Since $D_X N_A$ is the normal parts of $\bar{\nabla}_X N_A$ to M it is expressed as a linear combination of N_A, that is,

$$D_X N_A = \sum_{B=1}^{p} S_{AB}(X) N_B. \tag{2.3}$$

The ambient manifold $\bar{M}$ being of constant curvature c, the curvature tensor R, scalar curvature K and the normal curvature R^N are respectively given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\} + \sum_{A=1}^{p}\{g(H_A Y, Z)H_A X - g(H_A X, Z)H_A Y\}, \tag{2.4}$$

$$K = n(n-1)c + \sum_{A=1}^{p}(\mathrm{trace}\, H_A)^2 - \sum_{A=1}^{p}\mathrm{trace}\, H_A{}^2, \tag{2.5}$$

$$\begin{aligned} R^N(X, Y)N_A &= \sum_{B=1}^{p} g([H_A, H_B]X, Y)N_B \\ &= \sum_{B=1}^{p}\{(\nabla_X S_{AB})Y - (\nabla_Y S_{AB})X \\ &\quad + \sum_{C=1}^{p}(S_{AC}(Y)S_{CB}(X) - S_{AC}(X)S_{CB}(Y)\}N_B, \end{aligned} \tag{2.6}$$

where we put $[H_A, H_B]X = H_A H_B X - H_B H_A X$.

The mean curvature vector N is defined by

$$N = \sum_{A=1}^{p}(\mathrm{trace}\, H_A)N_A \tag{2.7}$$

and it is well known that N is independent of the choice of unit normal vectors N_A to M.

For some H_A, if there exists a function ρ_A such that

$$H_A X = \rho_A X, \tag{2.8}$$

at each point of M, we call M umbilical with respect to the normal N_A.

§ 3. Lemmas and proof of the theorem.

First of all we state the

LEMMA 1 [1]. *Let $a_1, \cdots, a_n$ and k be $n+1$ $(n \geqq 2)$ real numbers satisfying*

$$\sum_{i=1}^{n} a_i^2 + k < \frac{1}{n-1}\left(\sum_{i=1}^{n} a_i\right)^2, \tag{3.1}$$

then, for any pair of i, j $(i \neq j)$, we have

$$k<2a_ia_j.$$

Now let the submanifold M satisfy the condition of the theorem. Then by the assumption M has no minimal point and so we can choose the first nomal vector N_1 to M in the direction of the mean curvature vector N at each point. From the definition of the mean curvature vector, it follows that

$$\text{trace } H_A=0, \quad A=2, 3, \cdots, p. \tag{3.2}$$

Let $E_1, \cdots, E_n$ be mutually orthonormal eigenvectors of the second fundamental tensor H_1 and $a_1, \cdots, a_n$ corresponding eigenvalues to $E_1, \cdots, E_n$ respectively. Then we have

LEMMA 2. *In an n-dimensional submanifold M of a Riemannian manifold of constant curvature c, if the second fundamental tensors H_A satisfy (1.2) at a point $P\in M$, then the sectional curvature $R(i, j)$ for the plane section spanned by E_i and E_j is positive.*

Proof. Denoting components of H_A $(A=2, 3, \cdots, p)$ by $\lambda_{ji}{}^{A}$ we have, from (1.2) and (3.2),

$$\frac{1}{n-1}(\sum_{i=1}^{n}a_i)^2+2\delta>\sum_{i=1}^{n}a_i^2+\sum_{A=2}^{p}\sum_{i,k=1}^{n}(\lambda_{ik}{}^{A}\lambda_{ik}{}^{A}).$$

Applying Lemma 1 to the last equation, we get

$$\begin{aligned}2a_ia_j&>\sum_{A=2}^{p}\sum_{i,k=1}^{n}(\lambda_{ik}{}^{A}\lambda_{ik}{}^{A})-2\delta\\ &\geqq\sum_{A=2}^{p}\{(\lambda_{ii}{}^{A})^2+2(\lambda_{ij}{}^{A})^2+(\lambda_{jj}{}^{A})^2\}-2\delta\\ &\geqq2\sum_{A=2}^{p}\{|\lambda_{ii}{}^{A}\lambda_{jj}{}^{A}|+(\lambda_{ij}{}^{A})^2\}-2\delta,\end{aligned}$$

and hence,

$$a_ia_j>\sum_{A=2}^{p}\{|\lambda_{ii}{}^{A}\lambda_{jj}{}^{A}|+(\lambda_{ij}{}^{A})^2\}-\delta. \tag{3.3}$$

On the other hand, by (2.4), the sectional curvature $R(i, j)$ is given by

$$R(i. j)=g(R(E_i, E_j)E_j, E_i)=c+a_ia_j+\sum_{A=2}^{p}\{\lambda_{ii}{}^{A}\lambda_{jj}{}^{A}-(\lambda_{ij}{}^{A})^2\}. \tag{3.4}$$

Combining (3.3) and (3.4), we find

$$R(i, j)>c+\sum_{A=2}^{p}\{|\lambda_{ii}{}^{A}\lambda_{jj}{}^{A}|+(\lambda_{ij}{}^{A})^2+\lambda_{ii}{}^{A}\lambda_{jj}{}^{A}-(\lambda_{ij}{}^{A})^2\}-\delta\geqq c-\delta>0.$$

This completes the proof.

LEMMA 3. *Under the same assumptions of the theorem, M is compact.*

Proof. By Myers' theorem it is sufficient for us to prove that the Ricci tensor $\mathrm{Ric}(X, X)$ of M is greater than some positive number. So we compute $\mathrm{Ric}(X, X)$. Let E_i be a unit eigenvector of H_1 corresponding to the eigenvalue a_i. Then, putting $X=\sum_{j=1}^{n} x_j E_j$, $\sum_{j=1}^{n}(x^j)^2=1$, the sectional curvature for the plane section spanned by X and E_i is

$$g(R(X, E_i)E_i, X)=c\{\sum_{j=1}^{n}(x^j)^2-(x^i)^2\}+a_i\sum_{j=1}^{n}a_j(x^j)^2-(a_ix^i)^2 + \sum_{A=2}^{p}\{\lambda_{ii}{}^A\sum_{j,k=1}^{n}\lambda_{jk}{}^A x^j x^k-(\sum_{j=1}^{n}\lambda_{ji}{}^A x^j)^2\}.$$

Thus the Ricci tensor $\mathrm{Ric}(X, X)$ becomes

$$\mathrm{Ric}(X, X)=\sum_{i=1}^{n} g(R(X, E_i)E_i, X)$$

$$=(n-1)c+\sum_{i=1}^{n}\{a_ia_1(x^1)^2+\cdots+\widehat{a_i^2(x^i)^2}+\cdots+a_ia_n(x^n)^2\}$$

$$-\sum_{A=2}^{p}\sum_{i=1}^{n}(\sum_{j=1}^{n}\lambda_{ji}{}^A x^j)^2,$$

because of $\Sigma\lambda_{ii}{}^A=0$ for $A=2, \cdots, p$, where "$\wedge$" denotes a term which will be omitted. Substituting (3.3) into the last equation and making use of the fact that $\sum_{i=1}^{n}(x^i)^2=1$, we have

$$\mathrm{Ric}(X, X)>(n-1)(c-\delta)+\sum_{A=2}^{p}\{\frac{n-1}{2}\sum_{j,k=1}^{n}\lambda_{jk}{}^A\lambda_{jk}{}^A-\sum_{i=1}^{n}(\sum_{j=1}^{n}\lambda_{ji}{}^A x^j)^2\}$$

$$\geqq(n-1)(c-\delta)+\sum_{A=1}^{p}\{\frac{n-1}{2}\sum_{j,k=1}^{n}\lambda_{jk}{}^A\lambda_{jk}{}^A-\sum_{j,i=1}^{n}(\lambda_{ji}{}^A\lambda_{ji}{}^A)\sum_{k=1}^{n}(x^k)^2\}$$

$$=(n-1)(c-\delta)+\frac{n-3}{2}\sum_{A=2}^{p}\sum_{j,k=1}^{n}\lambda_{jk}{}^A\lambda_{jk}{}^A$$

$$\geqq(n-1)(c-\delta)>0.$$

This complete the proof.

Proof of the theorem. Lef f be the square of the length of the second fundamental tensor with respect to N_1, that is,

$$f=\mathrm{trace}\ H_1^2.$$

The Laplacian for f is given by [3]

$$\frac{1}{2}\Delta f=\sum_{i<j}R(i, j)(a_i-a_j)^2+\|\nabla H_1\|^2\geqq 0.$$

Since Lemma 3 shows that M is compact, we can apply the Bochner's lemma and get $f=const.$ Thus, we have $\nabla H_1=0$, $a_1=a_2=\cdots=a_n\neq 0$, because of Lemma 2. Consequently M is umbilical with respect to the mean curvature direction.

REMARK. In the theorem if M is assumed to be compact, the inequality (1.2) can be replaced by

$$\sum_{A=1}^{p} \operatorname{trace} H_A{}^2 < \frac{1}{n-1} \sum_{A=1}^{p} (\operatorname{trace} H_A)^2 + 2c,$$

because the number δ is only used to show that M is compact.

Bibliography

[1] M. Okumura, *Submanifolds and a pinching problem on the second fundamental tensors*, Transactions of American Math. Soc. **178**(1973) 285-291.

[2] M. Okumura, *A pinching problem on the second fundamental tensors and submanifolds of a sphere*, Tôhoku Math. J. **25**(1973) 461-467.

[3] B. Smyth. *Submanifolds of constant mean curvature*, Mathematishe Annalen **205** (1973) 265-280.

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