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CODIMENSION 2 SUBMANIFOLDS WITH A QUASI-UMBILICAL NORMAL DIRECTION

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§1 Introduction.

Let M^n be an n-dimensional submanifold of an $(n+2)$-dimensional Riemannian manifold $M^{n+2}(\tilde{c})$ of constant curvature $\tilde{c}$. Then there exist two second fundamental tensors, say (h_{ji}) and (k_{ji}) w. r. t. two orthonormal normal vector fields ξ and $\xi^{\perp}$. It seems to be interesting to find out some precise relations between these two second fundamental tensors. In [6], one of the authors and K. Yano proved that if (h_{ji}) has only one eigenvalue and ξ is non-parallel (in the normal bundle), then (k_{ji}) has an eigenvalue of multiplicity $\geqq n-1$.

In this paper, we shall find out the precise form of (k_{ji}) whenever (h_{ji}) has an eigenvalue of multiplicity $n-1$. In particular, we shall prove that in generic case (k_{ji}) has an eigenvalue of multiplicity $\geqq n-3$ (Theorem 3). Some applications are given.

§2. Preliminaries.

Let M^n be an n-dimensional submanifold(**) of an $(n+2)$-dimensional space form $M^{n+2}(\tilde{c})$ of curvature $\tilde{c}$. Let $\tilde{g}$ and $\tilde{\nabla}$ be respectively the metric tensor of $M^{n+2}(\tilde{c})$ and the corresponding operator of covariant differentiation. Let g and ∇ be the induced metric, respectively connection, on M^n. The second fundamental form h of M^n is given by

$$(1)\qquad \tilde{\nabla}_X Y=\nabla_X Y+h(X,Y)$$

where X and Y are tangent vector fields on M^n. $h(X,Y)$ is a normal vector field on M^n which is symmetric on X and Y. For a normal vector field η on M^n we write

$$(2)\qquad \tilde{\nabla}_X\eta=-A_\eta(X)+D_X\eta$$

where $-A_\eta(X)$ and $D_X\eta$ denote respectively the tangential and normal components of $\tilde{\nabla}_X\eta$. A_η is the *second fundamental tensor* of M^n w. r. t. η and D the normal connection of M^n in $M^{n+2}(\tilde{c})$. We have

$$(3)\qquad \langle A_\eta(X), Y\rangle=\langle h(X,Y),\eta\rangle$$

where $\langle\ ,\ \rangle$ denotes the scalar product in $M^{n+2}(\tilde{c})$.

Let $\{x^k\}$ be a local coordinate system in M^n (indices k,j,i,h,t,s running over the range $\{1,2,\cdots,n\}$), and put $\partial_i=\partial/\partial x^i$. Let ξ and $\xi^{\perp}$ be two orthogonal normal sections of

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(**) Manifolds, mappings, functions, $\cdots$ are assumed to be sufficiently differentiable, and our discussions are restricted to submanifolds of dimension $n>3$.

M^n. Then

(4) $$D_j\xi = l_j\xi^{\perp}, \qquad D_j\xi^{\perp} = -l_j\xi,$$

where $D_j = D_{\partial j}$ and l_j is the *third fundamental tensor.* ξ and $\xi^{\perp}$ are said to be parallel or non-parallel according to the third fundamental tensor vanishes or never vanishes identically.

In the followng we'll denote the second fundamental tensor w. r. t. ξ and $\xi^{\perp}$ respectively by h_{ji} and k_{ji}. Then the *equations of Gauss, Codazzi* and *Ricci* are respectively:

(5) $$K_{kjih} = \tilde{c}(g_{kh}g_{ji} - g_{jh}g_{ki}) + h_{kh}h_{ji} - h_{jh}h_{ki} + k_{kh}k_{ji} - k_{jh}k_{ki};$$

(6) $$\nabla_k h_{ji} + l_j k_{ki} = \nabla_j h_{ki} + l_k k_{ji},$$

(7) $$\nabla_k k_{ji} - l_j h_{ki} = \nabla_j k_{ki} - l_k h_{ji};$$

(8) $$\nabla_j l_i - \nabla_i l_j = h_{it}k_j^{\ t} - h_{jt}k_i^{\ t},$$

where K_{kjih} is the *Riemann-Christoffel curvature tensor* of $M^n(K(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]})$, and $h_j^{\ i} = h_{jt}g^{ti}$, $k_j^{\ i} = k_{jt}g^{ti}$.

If the Ricci tensor S, whose components are given by $K_{ji} = K_{tji}{}^t$ where $K_{kji}{}^h = K_{kjit}g^{th}$, is proportional to the metric tensor g then M^n is said to be *Einstein*; in particular, if $S=0$ then M^n is said to be *Ricci flat*. Let W and Z be two linearly independent vectors at a point p of M^n and γ the plane section spanned by W and Z. Then the *sectional curvature* for γ is defined by

$$k(\gamma) = \frac{g(K(W,Z)Z, W)}{g(W,W)g(Z,Z) - g(W,Z)^2}.$$

Putting

$$L_{ji} = -\frac{K_{ji}}{n-2} + \frac{Kg_{ji}}{2(n-1)(n-2)},$$

where $K = K_{ji}g^{ji}$, *Weyl's conformal curvature tensor* is given by

$$C_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k^h g_{ji} - L_j^h g_{ki},$$

where δ_k^h are Kronecker deltas and $L_k^h = L_{kt}g^{th}$. If $C_{kji}{}^h = 0$ then M^n is said to be *conformally flat.*

If on M^n there exist two functions α, β and a unit vector field u_j such that

(9) $$h_{ji} = \alpha g_{ji} + \beta u_j u_i$$

then M^n is said to be *quasi-umbilical* w. r. t the normal direction ξ. If respectively identically $\alpha=0$, $\beta=0$ or $\alpha=\beta=0$ then M^n is said to be *cylindrical, umbilical* or *geodesic* w. r. t. ξ. A quasi-umbilical normal direction ξ which is not umbilical is said to be *properly quasi-umbilical*; in this case the tangent direction of M^n determined by u_j is called the *distinguished direction* w. r. t. ξ. M^n is said to be a *totally quasi-umbilical* submanifold of $M^{n+2}(\tilde{c})$ if M^n is quasi-umbilical w. r. t. 2 orthogonal normal directions [1].

In the following, we shall simply denote by $w_j /\!/ u_j$ (respectively $w_j \not\!/\!/ u_j$) that a vector field w_j is parallel (respectively not parallel) to u_j everywhere. In the same way we use the notation $w_j=0$ and $w_j\neq 0$.

§ 3. Submanifolds quasi-umbilical w. r. t. a non-parallel normal direction.

The main purpose of this paragraph is to find expressions for the second fundamental tensor k_{ji} w. r. t. $\xi^{\perp}$ when the second fundamental tensor h_{ji} w. r. t. ξ is given by (9)(*) and $l_j\neq 0$.

From (6) and (9) we have

$$\alpha_k g_{ji}+\beta_k u_j u_i+\beta u_i \nabla_k u_j+\beta u_j \nabla_k u_i+l_j k_{ki} \tag{10}$$
$$=\alpha_j g_{ki}+\beta_j u_k u_i+\beta u_i \nabla_j u_k+\beta u_k \nabla_j u_i+l_k k_{ji},$$

where $\alpha_k=\partial_k\alpha$, $\beta_k=\partial_k\beta$. Transvecting g^{ji} to (10) gives

$$l^t k_{kt}=-(n-1)\alpha_k-\beta_k+(\beta u)u_k+k_t{}^t l_k+\beta u^t \nabla_t u_k+\beta u_k \nabla^t u_t, \tag{11}$$

where $(\beta u)=\beta_t u^t$, $l^s=g^{st}l_t$, $k_t{}^t=k_{ts}g^{st}$, $\nabla^t=g^{ts}\nabla_s$.
Transvecting l^j to (10) gives

$$l^2 k_{ki}=(l\alpha)g_{ki}+(l\beta)u_k u_i+\beta u_i l^t \nabla_t u_k+\beta u_k l^t \nabla_t u_i \tag{12}$$
$$-\alpha_k l_i-(lu)\beta_k u_i-\beta u_i l^t \nabla_k u_t-\beta(lu)\nabla_k u_i+l_k l^t k_{it},$$

where $l^2=l_t l^t$, $(l\alpha)=l_t\alpha^t$, $(l\beta)=l_t\beta^t$, $(lu)=l_t u^t$. From (11) and (12) we obtain

$$l^2 k_{ki}=(l\alpha)g_{ki}+(l\beta)u_k u_i+k_t{}^t l_k l_i+[(\beta u)+\beta\nabla^t u_t]l_k u_i \tag{13}$$
$$-(n-1)l_k\alpha_i-\alpha_k l_i-l_k\beta_i-(lu)\beta_k u_i+\beta(u_k l^t+l_k u^t)\nabla_t u_i$$
$$+\beta u_i l^t(\nabla_t u_k-\nabla_k u_t)-\beta(lu)\nabla_k u_i.$$

Transvecting u^j to (10) gives

$$(lu)k_{ki}=(\alpha u)g_{ki}+(\beta u)u_k u_i-(\alpha_k+\beta_k)u_i \tag{14}$$
$$+\beta u_i u^t \nabla_t u_k+\beta u_k u^t \nabla_t u_i-\beta\nabla_k u_i+l_k u^t k_{ti},$$

where $(\alpha u)=\alpha_t u^t$. Transvecting u^i to (14) gives

$$(lu)u^t k_{kt}=-\alpha_k-\beta_k+[(\alpha u)+(\beta u)]u_k+\beta u^t \nabla_t u_k+k(u,u)l_k, \tag{15}$$

where $k(u,u)=k_{ts}u^t u^s$. From (14) and (15) we obtain

$$(lu)^2 k_{ki}=(\alpha u)(ul)g_{ki}+(\beta u)(ul)u_k u_i+k(u,u)l_k l_i \tag{16}$$
$$+[(\alpha u)+(\beta u)]l_k u_i-l_k\alpha_i-l_k\beta_i-(lu)\beta_k u_i-(lu)\alpha_k u_i$$

(*) It will be permanently assumed that ξ is a proper quasiumbilical section.

$$+\beta l_k u^t \nabla_t u_i + \beta(lu)(u_i u^t \nabla_t u_k + u_k u^t \nabla_t u_i)$$
$$-\beta(lu)\nabla_k u_i.$$

Taking difference of (15) and (16) we have

$$(17)\qquad [l^2-(lu)^2]k_{ki} = [(l\alpha)-(\alpha u)(ul)]g_{ki} + [(l\beta)-(\beta u)(ul)]u_k u_i$$
$$+[k_t{}^t - k(u,u)]l_k l_i + [\beta\nabla^t u_t - (\alpha u)]l_k u_i - (n-2)l_k \alpha_i$$
$$-\alpha_k l_i + (lu)\alpha_k u_i + \beta(u_k l^t \nabla_t u_i + u_i l^t \nabla_t u_k)$$
$$-\beta(lu)(u_i u^t \nabla_t u_k + u_k u^t \nabla_t u_i) - \beta u_i l^t \nabla_k u_t.$$

Since k_{ki} is symmetric we derive from (17) that

$$(18)\qquad \beta(lu)l^t\nabla_k u_t = [(n-3)l^2 + (lu)^2]\alpha_k + [(lu)\beta\nabla^t u_t - (lu)(u\alpha)$$
$$-(n-3)(l\alpha)]l_k + [l^2(\alpha u) - l^2\beta\nabla^t u_t + \beta l^t l^s \nabla_t u_s - (lu)(l\alpha)]u_k.$$

Thus (17) becomes

$$(19)\qquad (lu)[l^2-(lu)^2]k_{ki} = (lu)[(l\alpha)-(\alpha u)(ul)]g_{ki} + [(lu)(l\beta)$$
$$-(lu)^2(u\beta) + \beta l^t l^s \nabla_t u_s - (lu)(l\alpha) - l^2\beta\nabla^t u_t + (\alpha u)l^2]u_k u_i$$
$$+(lu)[k_t{}^t - k(u,u)]l_k l_i + (n-3)(l\alpha)l_k u_i - (n-3)l^2\alpha_k u_i$$
$$-(n-2)(lu)l_k\alpha_i - (lu)\alpha_k l_i + \beta(lu)(u_k l^t\nabla_t u_i + u_i l^t \nabla_t u_k)$$
$$-\beta(lu)^2(u_i u^t\nabla_t u_k + u_k u^t \nabla_t u_i).$$

Again using symmetry of k_{ki} we find that

$$(20)\qquad [l^2-(lu)^2]\alpha_i = [(l\alpha)-(lu)(u\alpha)]l_i + [l^2(\alpha u) - (l\alpha)(lu)]u_i.$$

From (19) and (20) follows

LEMMA 1. *Let* $h_{ji} = \alpha g_{ji} + \beta u_j u_i$. *If* $D_j\xi \equiv l_j\xi^\perp \neq 0$, $(lu) \equiv l_i u^i \neq 0$ and $l_j \not\parallel u_j$ *at a point* p, *then at* p *we have*

$$(21)\qquad k_{ji} = [l^2-(lu)^2]^{-1}[(l\alpha)-(\alpha u)(ul)]g_{ji} + (lu)^{-1}[l^2-(lu)^2]^{-2}\times$$
$$\{(lu)^4(u\beta) + (lu)^3[(l\alpha)-(l\beta)] + l^2(lu)^2[\beta\nabla^t u_t - (\alpha u) - (\beta u)]$$
$$+l^2(lu)[(l\beta) + (n-4)(l\alpha)] + [l^2-(lu)^2]\beta l^t l^s \nabla_t u_s$$
$$-l^4[\beta\nabla^t u_t + (n-4)(\alpha u)]\}u_j u_i + [l^2-(lu)^2]^{-1}\times$$
$$\{k_t{}^t - k(u,u) - (n-1)[l^2-(lu)^2]^{-1}[(l\alpha)-(lu)(u\alpha)]\}l_j l_i$$
$$+[l^2-(lu)^2]^{-2}[(l\alpha)(lu) - (u\alpha)l^2](l_j u_i + l_i u_j)$$

$$+\beta[l^2-(lu)^2]^{-1}\{u_j[l^t-(lu)u^t]\nabla_t u_i$$
$$+u_i[l^t-(lu)u^t]\nabla_t u_j\}.$$

Next we consider the case $(lu)=0$. Then from (13) and (15) we obtain

(22)
$$l^2k_{ki}=(l\alpha)g_{ki}+(l\beta)u_ku_i+[k_t{}^t-nk(u,u)]l_kl_i$$
$$+[\beta\nabla^t u_t-(n-1)(\alpha u)-(n-2)(\beta u)]l_ku_i-[(\alpha u)+\beta u)]u_kl_i$$
$$+(n-2)l_k\beta_i+\beta_kl_i-(n-2)\beta l_ku^t\nabla_t u_i-\beta l_iu^t\nabla_t u_k$$
$$-\beta u_il^t\nabla_k u_t+\beta(u_kl^t\nabla_t u_i+u_il^t\nabla_t u_k).$$

From (22) it follows by the symmetry of k_{ki} that

(23)
$$\beta l^t\nabla_k u_t=\beta u_kl^tu^s\nabla_s u_t+[\beta\nabla^t u_t-(n-2)(\alpha u)]l_k$$

and

(24)
$$(n-3)\beta l^2u^t\nabla_t u_k=[\beta l^2\nabla^t u_t-(n-2)l^2(\alpha u)-(n-3)l^2(\beta u)$$
$$-\beta l^tl^s\nabla_s u_t]u_k+(n-3)[\beta l^sl^t\nabla_t u_s-(\beta l)]l_k+(n-3)l^2\beta_k.$$

Making use of these expressions in (22) we obtain

LEMMA 2. *Let* $h_{ji}=\alpha g_{ji}+\beta u_ju_i$. If $D_j\xi=l_j\xi^\perp\neq 0$ and $l_j\perp u_j$ *at a point* p, *then at* p *we have*

(25)
$$k_{ji}=l^{-2}\{(l\alpha)g_{ji}+[(l\beta)-\beta l^tu^s\nabla_s u_t]u_ju_i+l^{-2}[l^2k_t{}^t$$
$$-nl^2k(u,u)-(n-1)\beta u^tl^s\nabla_t u_s+(n-1)(l\beta)]l_jl_i$$
$$-(\alpha u)(u_jl_i+l_ju_i)+\beta(u_jl^t\nabla_t u_i+u_il^t\nabla_t u_j)\}.$$

From Lemmas 1 and 2 we immediately have

THEOREM 3. *Let* $h_{ji}=\alpha g_{ji}+\beta u_ju_i$. *If* $D_j\xi=l_j\xi^\perp\neq 0$ *and* $l_j\not\times u_j$ *at a point* p, *then at* p k_{ji} *has an eigenvalue of multiplicity* $\geqq n-3$.

Finally, in case $l_j=(lu)u_j$, we derive from (17) that

(26)
$$(n-2)\alpha_i=[(lu)k_t{}^t-(lu)k(u,u)+\beta\nabla^t u_t-(\alpha u)]u_i.$$

Substitution of (26) in (13) leads to the following:

THEOREM 4. *Let* $h_{ji}=\alpha g_{ji}+\beta u_ju_i$. *If* $D_j\xi=l_j\xi^\perp\neq 0$ *and* $l_j\not\parallel u_j$ *at a point* p, *then at* p *we have*

(27)
$$k_{ji}=l^{-2}\{(l\alpha)g_{ji}+[l^2k(u,u)+2(l\beta)-(l\alpha)]u_ju_i$$
$$-(lu)(\beta_ju_i+u_j\beta_i)-\beta(lu)\nabla_j u_i+\beta(lu)(u_iu^t\nabla_t u_j+2u_ju^t\nabla_t u_i)\}.$$

§ 4. Locus of certain submanifolds.

In this section we prove that for a submanifold M^n of $M^{n+2}(\tilde{c})$ which is quasi-umbilical w. r. t. a normal direction ξ the distribution defined by $u_j dx^j = l_j dx^j = 0$ is involutive, and study the corresponding integral submanifolds.

THEOREM 5. *Let M^n be a submanifold of $M^{n+2}(\tilde{c})$ with $h_{ji} = \alpha g_{ji} + \beta u_j u_i$.*
(i) If $l_j \neq 0$ and $l_j \not\parallel u_j$, then M^n is foliated by hypersurfaces which are umbilical w. r. t ξ and a normal direction spanned by $\xi^\perp$ and u_j.
(ii) If $l_j \neq 0$ and $l_j \parallel u_j$, then M^n is foliated by submanifolds of codimension 2 which are umbilical w. r. t. ξ, $\xi^\perp$ and u_j.
(iii) If $l_j = 0$, then M^n is foliated by hypersurfaces which are umbilical w. r. t. ξ and u_j.
In all three cases ξ is parallel in the normal bundle of the leaves of M^n in $M^{n+2}(\tilde{c})$.

Proof. (i) In case $l_j \neq (lu) u_j$, it follows by straightforward calculations from (8), (9), (14), (21) and (25) that the distribution $u_j dx^j = l_j dx^j = 0$ is integrable. From (21) and (25) we see that the corresponding integral submanifolds M^{n-2} of M^n are umbilical w. r. t. $\xi^\perp$. Consequently, by (13), they are also umbilical w. r. t. u_j.

(ii) In case $l_j \neq 0$ and $l_j = (lu) u_j$, it follows from (27) by the symmetry of k_{ki} that the distribution $u_j dx^j = 0$ is integrable. Also from (27) it is clear that the corresponding integral submanifolds M^{n-1} of M^n are umbilical w. r. t. a normal direction spanned by $\xi^\perp$ and u_j.

(iii) In case $l_j = 0$, we find by transvecting u^i to (10) that

$$\beta(\nabla_k u_j - \nabla_j u_k) = (\alpha_j + \beta_j) u_k - (\alpha_k + \beta_k) u_j, \tag{28}$$

which implies that the distribution $u_j dx^j = 0$ is integrable. Since (14) actually becomes

$$\beta \nabla_k u_i = (\alpha u) g_{ki} + (\beta u) u_k u_i - (\alpha_k + \beta_k) u_i + \beta (u_i u^t \nabla_t u_k + u_k u^t \nabla_t u_i) \tag{29}$$

the corresponding integral submanifolds M^{n-1} of M^n are umbilical w. r. t. u_j.

Of course in all cases the integral submanifolds under consideration are umbilical w. r. t. ξ.

Finally, let D' denote the normal connection of $M^{n-1}(M^{n-2})$ in $M^{n+2}(\tilde{c})$. Then, since from (2) and (9) it follows that

$$\tilde{\nabla}_{X'} \xi = -\alpha X' + D_{X'} \xi \tag{30}$$

where X' is a tangent vector field on $M^{n-1}(M^{n-2})$, we have

$$D'_{X'} \xi = D_{X'} \xi. \tag{31}$$

From (31) and (4) we see that ξ is parallel on $M^{n-1}(M^{n-2})$.

REMARKS.

1. From Theorem 5 and [4] we conclude that the leaves $M^{n-1}(M^{n-2})$ of M^n are *spheri-*

cal submanifolds, i.e. lie in a hypersphere of $M^{n+2}(\tilde{c})$.

2. For results on submanifolds M^n of a Riemannian manifold N^m which are *umbilical* w.r.t. an $(m-n-1)$-*dimensional normal subbundle*, we refer to [6, 7].

we recall that a vector field X on a Riemanian manifold V is called a *conformal* (in particular a *Killing*) vector field if $\mathcal{L}_X g=2\rho g$ ($\mathcal{L}_X g=0$), where g is the metric tensor of V, $\mathcal{L}$ Lie differentiation, ρ a function. The field of 1-forms associated with a conformal (Killing) vector field will also said to be conformal (Killing). We have the following:

LEMMA 6. *The leaves of the foliation on a Riemannian manifold associated with a conformal (respectively Killing) field of 1-forms are totally umbilical (respectively totally geodesic) hypersurfaces.*

From Lemma 6 and Theorems 4 and 5 (i), we obtain

PROPOSITION 7. *Let M^n be a submanifold of $M^{n+2}(\tilde{c})$ with $h_{ij}=\alpha g_{ji}+\beta u_j u_i$ where u_j is conformal. If $l_j\neq 0$ and $l_j \not\parallel u_j$, then M^n is a locus of $(n-1)$-spheres.*

§ 5. $h_{ji}=\alpha g_{ji}+\beta u_j u_i$ with $\nabla_k u_j=0$.

The main purpose of this paragraph is to find expressions for the second fundamental tensor k_{ji} w.r.t. $\xi^{\perp}$ when M^n is quasi-umbilical w.r.t. ξ and the corresponding distinguished section is parallel.

In case ξ is non-parallel we obtain from the equation (6) of Codazzi, making use of the symmetry of k_{ki} and the fact that since u_j is parallel the sectional curvatures for plane sections containing u_j vanish, the following.

LEMMA 8. *Let $h_{ji}=\alpha g_{ji}+\beta u_j u_i$ with $\nabla_k u_i=0$. If $l_j\neq 0$ at a point p, then at p we have*

$$k_{ji}=a g_{ji}+b u_j u_i+c l_j l_i \tag{32}$$

whereby a, b, c satisfy one of the following cases

$$a=(l\alpha)l^{-2},\quad b=k_t{}^t-n(l\alpha)l^{-2},\quad c=0; \tag{33}$$

$$a=-b=(l\alpha)l^{-2},\quad c=k_t{}^t l^{-2}-(n-1)(l\alpha)l^{-4}. \tag{34}$$

In particular k_{ji} has an eigenvalue of multiplicity $\geqq n-2$.

From Lemma 8 we immediately have

THEOREM 9. *A submanifold M^n of $M^{n+2}(\tilde{c})$ for which $h_{ij}=\alpha g_{ji}+\beta u_j u_i$ with $\nabla_k u_j=0$ and $l_j\neq 0$ is a locus of $(n-1)$-dimensional totally quasi-umbilical submanifolds.*

REMARK. From Theorem 9 and [5] it follows that a submanifold M^n of $M^{n+2}(\tilde{c})$ with $n>4$ which is quasi-umbilical w.r.t. a non-parallel normal direction with parallel corresponding distinguished direction is a locus of $(n-1)$-dimensional conformally flat spaces.

Next we consider the case where ξ is parallel.

LEMMA 10. *Let $h_{ji}=\alpha g_{ji}+\beta u_j u_i$ with $\nabla_k u_i=0$. If $l_j=0$ at a point p, then at p we have either*

$$k_{ji}u^j v^i=0 \text{ for any vector field } v^k, \tag{35}$$

or

$$k_{ji}=\lambda g_{ji}+\mu u_j u_i \tag{36}$$

where μ is a non-zero function and

$$\lambda=-[\tilde{c}+(\alpha+\beta)\alpha]\mu^{-1}. \tag{37}$$

Proof. If ξ is parallel then the normal connection of M^n in $M^{n+2}(\tilde{c})$ is flat. Consequently the second fundamental tensors can be diagonalized simultaneously, i. e. by an appropriate choice of basis for tangent space of M^n we have

$$h_{ji}=\begin{pmatrix} \alpha+\beta & & & & 0 \\ & \alpha & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \alpha \end{pmatrix}, \qquad k_{ji}=\begin{pmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \lambda_n \end{pmatrix}. \tag{38}$$

Since the sectional curvature for any plane section containing u_j is zero we have

$$(\alpha+\beta)\alpha+\lambda_1\lambda_\tau+\tilde{c}=0, \tag{39}$$

where $\tau\in\{2, 3, \cdots, n\}$. If $\lambda_1=0$, then clearly (35) holds. If $\lambda_1\neq 0$, then it follows from (39) that

$$\lambda_\tau=-[\tilde{c}+(\alpha+\beta)\alpha]\lambda_1^{-1}. \tag{40}$$

Still in case ξ is parallel, we derive from (39) that for a submanifold M^n of E^{n+2} if $\lambda_1=0$ then either $\alpha=0$ or $\alpha+\beta=0$. Therefore by the parallelism of u_j we conclude using [9], that M^n is a product submanifold $M^1\times M^{n-1}\subset E^2\times E^n$ or $E^1\times M^{n-1}\subset E^1\times E^{n+1}$. In the first case M^{n-1} is an arbitrary hypersurface of E^n. In the second case M^{n-1} is a submanifold of E^{n+1} for which ξ determines a parallel umbilical normal direction; consequently $M^{n-1}\subset S^n\subset E^{n+1}$, where S^n denotes a hypersphere of E^{n+1}. If $\lambda_1\neq 0$ it follows from Lemma 10 that M^n is a totally quasi-umbilical submanifold with 1 distinguished direction. By the parallelism of u_j we conclude that M^n is a product submanifold $M^1\times S^{n-1}\subset E^2\times E^n$. Summarizing we formulate

THEOREM 11. *A submanifold M^n of E^{n+2} for which $h_{ji}=\alpha g_{ji}+\beta u_j u_i$ with $\nabla_k u_j=0$ and $l_j=0$ is one of the following product submanifolds:*

$$M^1\times M^{n-1}\subset E^2\times E^n, \qquad E^1\times M^{n-1}\subset E^1\times S^n\subset E^{n+2}.$$

§6. α constant or β constant or ξ isoperimetric.

Throughout this paragraph we assume that ξ is non-parallel and u_j is parallel.

PROPOSITION 12. *Let* $h_{ji}=\alpha g_{ji}+\beta u_j u_i$ *with* $\nabla_k u_i=0$. *If* $l_j\neq 0$ *at a point* p *and* α *is constant, then at* p, k_{ji} *is one of the following*:

$$k_{ji}=k_t{}^t u_j u_i; \tag{41}$$

$$k_{ji}=l^{-2}k_t{}^t l_j l_i, \quad (l_j\perp u_j). \tag{42}$$

In particular if $l_j\neq 0$ *and* $\alpha=0$, *then the ambient space form is necessarily euclidean and* M^n *is a product submanifold* $M^1\times E^{n-1}\subset E^3\times E^{n-1}$.

Proof. If α is constant, then (33) and (34) reduce to (41) and (42); in case (42), $l_j\perp u_j$ follows from the fact that actually the normal connection of M^n is flat [2]. In both cases we clearly have

$$(\alpha+\beta)\alpha+\tilde{c}=0. \tag{43}$$

Therefore if $\alpha=0$ we must have $\tilde{c}=0$. From [9] and the assumption that ξ is cylindrical and non-parallel it follows that M^n is the product submanifold of a space curve and E^{n-1}.

REMARK. In particular Proposition 12 asserts that if ξ is in the *mean curvature direction* of M^n, then $\xi^\perp$ is geodesic. Using [7] this implies that ξ must be cylindrical.

We recall that ξ is by definition *isoperimetric* if $h_t{}^t$ is constant [1]. From Lemma 8 and formula (13) we obtain after some calculations

PROPOSITION 13. *Let* $h_{ji}=\alpha g_{ji}+\beta u_j u_i$ *with* $\nabla_k u_j=0$. *If* $l_j\neq 0$ *at a point* p *and either* β *is constant or* $n\alpha+\beta$ *is constant, then at* p, k_{ji} *is one of the following*:

$$k_{ji}=(l\alpha)l^{-2}g_{ji}+[k_t{}^t-nl^{-2}(l\alpha)]u_j u_i; \tag{44}$$

$$k_{ji}=l^{-2}k_t{}^t l_j l_i, \quad (l_j\perp u_j). \tag{45}$$

From [5] and Propositions 12 and 13 we immediately have

THEOREM 14. *A submanifold* M^n of $M^{n+2}(\tilde{c})$ *for which* $h_{ji}=\alpha g_{ji}+\beta u_j u_i$, $\nabla_k u_i=0$, $l_j\neq 0$ *and* (i) α *is constant or* (ii) β *is constant or* (iii) $n\alpha+\beta$ *is constant, is conformally flat*; *in particular if* ξ *is cylindrical, then* M^n *is flat*.

§ 7. Einstein and conformally flat submanifolds.

In this section we study the submanifolds M^n which are quasi-umbilical w. r. t. a normal direction with parallel corresponding distinguished section and which are Einstein or conformally flat.

THEOREM 15. *If a submanifold* M^n *of* $M^{n+2}(\tilde{c})$ *for which* $h_{ji}=\alpha g_{ji}+\beta u_j u_i$ *with* $\nabla_j u_i=0$ *is Einstein, then necessarily* $\tilde{c}=0$ *and* M^n *is flat*.

Proof. By the parallelism of u_j, the Ricci curvature of M^n w.r.t. u_j vanishes. Consequently an Einstein submanifold M^n is Ricci flat.

If in case $l_j=0$ (35) holds, then we obtain by a straight forward calculation from $S=0$ that

$$\lambda_\tau^2 - k_t^t \lambda_\tau - (n-2)\alpha^2 = 0 \tag{46}$$

and moreover $\alpha+\beta=0$ or $\alpha=0$. From this we see that M^n is cylindrical w.r.t. $\xi^\perp$. If in case $l_j\neq 0$ (34) holds with $l_j \not\parallel u_j$ and $c\neq 0$, then the Ricci tensor of M^n is found to be

$$\begin{aligned} K_{ji} = & \{(n-1)\tilde{c} + (n-1)\alpha^2 + \alpha\beta + (l\alpha)l^{-2}[k_t^t - (l\alpha)l^{-2}]\} g_{ji} \\ & + \{(n-2)\alpha\beta - (l\alpha)l^{-2}[k_t^t - (l\alpha)l^{-2}]\} u_j u_i \\ & + l^{-2}[k_t^t - (n-1)l^{-2}(l\alpha)][k_t^t - (n-3)l^{-2}(l\alpha)] l_j l_i \\ & + l^{-4}(lu)(l\alpha)[k_t^t - (n-1)l^{-2}(l\alpha)](u_j l_i + u_i l_j). \end{aligned} \tag{47}$$

Consequently $S=0$ implies that

$$k_{ji} = l^{-2}(l\alpha)(g_{ji} - u_j u_i - 2l^{-2} l_j l_i), \quad l_j \perp u_j. \tag{48}$$

Then comparing the Ricci curvatures w.r.t. l_j and any vector field orthogonal to u_j and l_j it follows that in this case $(l\alpha)=0$.

From the preceding observations and Lemmas 8 and 10 it follows that a Ricci flat submanifold M^n is totally quasi-umbilical; more precisely we find after some additional calculations that M^n is totally cylindrical.

Since a totally quasi-umbilical Einstein submanifold of a space form is also a space form, we conclude that actually M^n is flat. Moreover by the cylindricity of ξ and $\xi^\perp$ the ambiant space form must be euclidean.

THEOREM 16. *A conformally flat submanifold M^n of $M^{n+2}(\tilde{c})$ for which $h_{ji}=\alpha g_{ji}+\beta u_j u_i$ with $\nabla_j u_i=0$ is a locus of $(n-1)$-dimensional space forms.*

Proof. From [8] it is known that if M^n is conformally flat then $\xi^\perp$ is quasi-umbilical. From Lemmas 8 and 10 this implies that either the distinguished direction w.r.t. $\xi^\perp$ is determined by u_j or $\xi^\perp$ is cylindrical with corresponding distinguished direction $l_j \perp u_j$. By the parallelism of u_j the integral submanifolds M^{n-1} of the distribution $u_i dx^i=0$ are geodesic w.r.t. their normal direction determined by u_j. Since moreover they are umbilical w.r.t. ξ and umbilical or cylindrical w.r.t. $\xi^\perp$, the submanifolds M^{n-1} have constant sectional curvature.

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