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ON THE APPROXIMATION OF A MINIMAL SURFACE

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The problem of Plateau is concerned with the existence of a minimal surface bounded by a given Jordan curve in n-dimensional Euclidean space $E^n(n\geqq 2)$. While this problem was solved by several mathematicians such as Riemann, Schwarz and Weierstrass during the 19th century, the complete and satisfactory solution to the problem was given only in 1930–1931, independently, by J. Douglas [3, 4] and T. Radó [6].

J. Douglas' method was independent of the Riemann mapping theorem. And by restricting himself to harmonic surfaces only and by considering a functional which is equivalent to the Dirichlet integral, he was able to treat the problem as a one-dimensional problem. On the other hand, the method of T. Radó was a continuation of the classical idea based on the least area characterization of a minimal surface and the theory of conformal mapping. Their work on the problem was later greatly simplified by R. Courant [2] and L. Tonelli [11] both by widening the class of admissible surfaces to those which are continuous and piecewise smooth. In 1970 the author [1] treated this problem as a one-dimensional variational problem and was able to show the existence of a soluti on to the problem of Plateau. In this note we study a computational device for approximating a minimal surface under the assumption that the given Jordan curve is sufficiently nice.

We begin with some preliminaries.

Let B denote the open unit disc $\{(u,v): u^2+v^2<1\}$ in E^2. The boundary of B will be denoted by ∂B and the closure of B by $\bar{B}$. For a surface ζ: $\bar{B}\rightarrow E^n$, we use the notation

$$E=\zeta_u\cdot\zeta_u=\zeta_u{}^2,\quad F=\zeta_u\cdot\zeta_v=\zeta_u\zeta_v,\quad G=\zeta_v\cdot\zeta_v=\zeta_v{}^2$$

We say that a surface $\zeta=(\zeta^1,\zeta^2,\cdots\zeta^n)$ in E^n is a minimal surface if the following conditions are satisfied:

(M1) ζ^j, $j=1,2,\cdots,n$, are continuous on $\bar{B}$ and harmonic in B

(M2) ζ satisfies the relations $E=G$ and $F=0$ in B.

Thus the problem of Plateau is formulated precisely as follows: given a Jordan curve Γ in E^n, we wish to prove the existence of a surface ζ which satisfies conditions (M1) and (M2) of above and, in addition, $\zeta|_{\partial B}$ is a topological representaion of Γ.

Now let Δ be the class of all the topological representations of the given Jordan curve Γ. For any $g=(g^1,g^2,\cdots g^n)$ in Δ, if we expand g^i in a Fourier series

$$g^i(\theta)\sim\frac{a_0{}^i}{2}+\sum_{k=1}^{\infty}(a_k{}^i\cos k\theta+b_k{}^i\sin k\theta)$$

then the harmonic function h^i determined by g^i is given by

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$$h^i(u, v)=\frac{a_0^i}{2}+\sum_{k=1}^{\infty} r^k(a_k^i \cos k\theta+b_k^i \sin k\theta)$$

where $u=r\cos\theta$ and $v=r\sin\theta$. And the harmonic surface with boundary value g is given by $h=(h^1, h^2, \cdots h^n)$.

Let us denote

$$a_k=(a_k^1,\ a_k^2, \cdots a_k^n)$$

$$b_k=(b_k^1,\ b_k^2, \cdots b_k^n)$$

$$a_k^2=a_k\cdot a_k,\quad b_k^2=b_k\cdot b_k,\quad a_k b_k=a_k\cdot b_k$$

and

$$g(\theta)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k \cos k\theta+b_k \sin k\theta)$$

Then the Dirichlet integral of the harmonic surface h is given by

$$\iint_B (h_u^2+h_v^2)\,dudv$$

$$=\int_0^{2\pi}\int_0^1 \left(h_r^2+\frac{1}{r^2}h_\theta^2\right) r\,dr\,d\theta$$

$$=\pi\sum_{k=1}^{\infty} k(a_k^2+b_k^2)$$

Since the value of the integral is completely determined by g, we write

$$J(g)=\pi\sum_{k=1}^{\infty} k(a_k^2+b_k^2)$$

It was shown in [1] that there exists a representation g in Δ which minimizes J and if h is the corresponding harmonic surface, then it satisfies the condition

$$rh_r(r,\theta)\,h_\theta(r,\theta)=0$$

in B which is equivalent to the condition $E=G$ and $F=0$. Hence h is a minimal surface bounded by Γ.

We now assume that the minimizing representation g is of class C^2. Let us denote by $G(g)$ the Fourier series of g.

$$G(g)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k \cos k\theta+b_k \sin k\theta)$$

and by $\bar{G}(g)$ the conjugate series

$$\bar{G}(g)=\sum_{k=1}^{\infty}(a_k \sin k\theta-b_k \cos k\theta).$$

The series obtained by the formal termwise differentiation will be denoted by $G'(g)$ and $\bar{G}'(g)$.

Then it is immediate from [9] that we have the following:

(1). $G(g)$ and $\bar{G}(g)$ converge uniformly and absolutely to the functions g and $\bar{g}$, respectively, where $\bar{g}$ is given by

$$\bar{g}(\theta)=-\frac{1}{\pi}\int_0^{\pi}\frac{g(\theta+t)-g(\theta-t)}{2\tan t/2}dt.$$

Moreover, $\bar{g}\in L_2\ (0,2\pi)$ and $\bar{G}(g)=G(\bar{g})$.

(2). $G'(g)$ and $\bar{G}'(g)$ converge uniformly and absolutely to g' and $\bar{g}'$, respectively, and thus $\bar{g}'$ is continuous on $[0,2\pi]$. Furthermore $G'(g)=G(g')$ and $\bar{G}'(g)=G(\bar{g}')$.

If we denote h and $\bar{h}$ the harmonic surfaces determined by $G(g)$ and $\bar{G}(g)$, respectively, then $\bar{h}$ is conjugate to h and

(3). $\lim\limits_{r\to 1}\bar{h}(r,\theta)=\bar{g}(\theta)$.

(4). For any θ_0, $h_\theta(r,\theta)$ converges to $g'(\theta_0)$ as $(r,\theta)\to(1,\theta_0)$ along any path not touching the circle ∂B.

(5). $\lim\limits_{r\to 1} rh_r(r,\theta)=\lim\limits_{r\to 1}\bar{h}_\theta(r,\theta)=\bar{g}'(\theta)$.

From (4) and (5) above we see that the harmonic function rh_rh_θ is determined by $g'\bar{g}'$ and, by the maximum principle, we also note that $g'(\theta)\bar{g}'(\theta)=0$ for all θ in $[0,2\pi]$ implies $rh_r(r,\theta)h_\theta(r,\theta)=0$ in B.

This gives us the following theorem.

THEOREM 1. *If a Jordan curve Γ in E^n admits a topological representation g which is of class C^2 and if g has the property that $g'(\theta)\bar{g}'(\theta)=0$ on $[0,2\pi]$, then the harmonic surface h determined by g is a minimal surface with boundary Γ.*

Since, by (2), $G(g')$ and $G(\bar{g}')$ converge absolutely, $G(g'\bar{g}')$ can be obtained by a formal multiplication of $G(g')$ and $G(\bar{g}')$ using Cauchy's rule and the resulting series also converges absolutely:

$$g'(\theta)\,\bar{g}'(\theta)=\sum_{m=2}^{\infty}(A_m\cos m\theta+B_m\sin m\theta)$$

where

$$A_m=\frac{1}{2}\sum_{k+l=m}kl(a_kb_l+b_ka_l)$$

$$B_m=\frac{1}{2}\sum_{k+l=m}kl(b_kb_l-a_ka_l)$$

Thus we see, by the above theorem, that if $A_m=B_m=0$ for all $m=2,3,\cdots$, then the harmonic surface h determined by g is a minimal surface.

Now suppose λ is an arbitrary real analytic function defined on $[0,2\pi]$ which is periodic of period 2π. And let $\varphi_\varepsilon(\theta)=\theta+\varepsilon\lambda(\theta)$ for any real ε. Then we have the following result which is of fundamental importance for our later work:

THEOREM 2. *If g is a topological representation of Γ which is of class C^2, then*

$$J(g\circ\varphi_\varepsilon)-J(g)<2\varepsilon\int_0^{2\pi}g'(\theta)\bar{g}'(\theta)\lambda(\theta)d\theta+\varepsilon^2R$$

where $|R|<cJ(g)$ and c depends only on $|\lambda(\theta)|$ and $|\lambda'(\theta)|$.

Proof of this theorem is a straight forward computation of the difference quotient

$$\frac{1}{\varepsilon}[J(g\circ\varphi_\varepsilon)-J(g)]$$

using the Fourier series of g' and $\bar{g}'$, thus we omit the detail here.

We are now in a position to proceed to find a computational method for approximating a minimal surface. This method will lead us to a surface which we call an "approximate minimal surface" in the sense that rh_rh_θ is small in absolute value.

To this end, we shall limit ourselves to a special class of Jordan curves: namely, Jordan curves which admit of a C^2-representation.

Suppose Γ is a given Jordan curve and $g_0(\theta)=(g_0^1(\theta),\ g_0^2(\theta),\cdots g_0^n(\theta))$ is a representation of Γ which is of class C^2. Let N_0 be a fixed positive integer which is so large that the tail end of the series $J(g_0)$ is neglible and consider variations of the form

$$\lambda(\theta)=\sum_{m=2}^{N_0}(\alpha_m\cos m\theta+\beta_m\sin m\theta)$$

where

$$(\alpha,\beta)=(\alpha_2,\alpha_3,\cdots\alpha_{N_0},\beta_2,\beta_3,\cdots\beta_{N_0})$$

is subject to the condition $\|(\alpha,\beta)\|=1$, i. e. (α,β) varies on the unit sphere of $E^{2(N_0-1)}$.

Note that as (α,β) varies subject to the above condition, both $|\lambda(\theta)|$ and $|\lambda'(\theta)|$ are uniformly bounded by some constant.

Now if a_k, b_k are the Fourier coefficients of g_0, then

$$g_0'(\theta)\bar{g}_0'(\theta)=\sum_{m=2}^{\infty}(A_m^0\cos m\theta+B_m^0\sin m\theta)$$

where A_m^0 and B_m^0 are given by Cauchy's multiplication rule as before.
And

$$2\int_0^{2\pi}g_0'(\theta)\bar{g}_0'(\theta)\lambda(\theta)d\theta=2\pi\sum_{m=2}^{N_0}(\alpha_mA_m^0+\beta_mB_m^0)$$

Thus the formula in Theorem 2 becomes

$$J(g_0\circ\varphi_\varepsilon)<J(g_0)+2\pi\varepsilon\sum_{m=2}^{N_0}(\alpha_mA_m^0+\beta_mB_m^0)+\varepsilon^2cJ(g_0)$$

where $\varphi_\varepsilon(\theta)=\theta+\varepsilon\lambda(\theta)$ and c is a constant depending only on the bounds of $|\lambda(\theta)|$ and $|\lambda'(\theta)|$ and thus independent of the choice of (α,β).

We now wish to find a direction in which the variation given by above formula takes on a maximum rate, that is, we wish to maxmize

$$\sum_{m=2}^{N_0} (\alpha_m A_m^0 + \beta_m B_m^0)$$

in absolute value subject to the condition $\|(\alpha, \beta)\| = 1$. But clearly this maximum rate is attained if we take

$$\alpha_m^0 = \frac{A_m^0}{\|(A^0, B^0)\|}, \quad \beta_m^0 = \frac{B_m^0}{\|(A^0, B^0)\|}$$

for $m = 2, 3, \cdots N_0$, where

$$(A^0, B^0) = (A_2^0, A_3^0, \cdots A_{N_0}^0, B_2^0, B_3^0, \cdots B_{N_0}^0).$$

Let

$$\lambda_0(\theta) = \sum_{m=2}^{N_0} (\alpha_m^0 \cos m\theta + \beta_m^0 \sin m\theta)$$

then

$$2\int_0^{2\pi} g_0'(\theta)\bar{g}_0'(\theta)\lambda_0(\theta)d\theta = 2\pi\|(A^0, B^0)\|$$

and thus if we put $g_1(\theta) = g_0(\theta + \varepsilon\lambda_0(\theta))$, then

$$J(g_1) < J(g_0) + 2\pi\varepsilon\|(A^0, B^0)\| + \varepsilon^2 cJ(g_0).$$

We now choose $M \geqq J(g_0)$ suitably large that, for

$$\varepsilon = -\frac{\|(A^0, B^0)\|}{Mc},$$

the function $\theta + \varepsilon\lambda_0(\theta)$ is biunique. Then

$$J(g_1) < J(g_0) - (2\pi - 1)\frac{\|(A^0, B^0)\|^2}{Mc}.$$

Since the Dirichlet integral is invariant under a conformal map, we may assume that, for three distinct points p_1, p_2, p_3 of ∂B, $g_1(p_i) = g_0(p_i)$, $i = 1, 2, 3$.

Now repeating the same argument to the representation g_1 and by induction, we get a sequence of representations $\{g_\nu\}$ of Γ with the following properties:

$$(1). \quad g_\nu'(\theta)\bar{g}_\nu'(\theta) = \sum_{m=2}^{\infty} (A_m^\nu \cos m\theta + B_m^\nu \sin m\theta),$$

where

$$A_m^\nu = \frac{1}{2}\sum_{k+l=m} kl(a_k^\nu b_l^\nu + b_k^\nu a_l^\nu)$$

$$B_m{}^\nu=\frac{1}{2}\sum_{k+l=m} kl(b_k{}^\nu b_l{}^\nu-a_k{}^\nu a_l{}^\nu)$$

with $a_k{}^\nu$, $b_k{}^\nu$ Fourier coefficient of g_ν.

(2). $$g_{\nu+1}(\theta)=g_\nu(\theta+\varepsilon\lambda_\nu(\theta))$$

where

$$\lambda_\nu(\theta)=\frac{1}{\|(A^\nu, B^\nu)\|}\sum_{m=2}^{N_0}(A_m{}^\nu \cos m\theta+B_m{}^\nu \sin m\theta)$$

and

$$\varepsilon=-\frac{\|(A^\nu, B^\nu)\|}{Mc}$$

(3). $$J(g_{\nu+1})<J(g_\nu)-(2\pi-1)\frac{\|(A^\nu, B^\nu)\|^2}{Mc}$$

Note that the choice of M and c can be made independent of ν because both $|\lambda_\nu(\theta)|$ and $|\lambda_\nu'(\theta)|$ are uniformly bounded. Moreover, as already noted, we may assume that the sequence $\{g_\nu\}$ has the three point property.

Now consider the sequence $\{J(g_\nu)\}$. This sequence is monotone decreasing and is bounded below. Thus it converges to some number so that $J(g_\nu)-J(g_{\nu+1})\to 0$ as $\nu\to\infty$. But from (3)

$$0\leqq(2\pi-1)\ \frac{\|(A^\nu, B^\nu)\|^2}{Mc}<J(g_\nu)-J(g_{\nu+1}).$$

Thus

$$\|(A^\nu, B^\nu)\|\to 0 \text{ as } \nu\to\infty$$

i. e.

$$\begin{matrix} A_m{}^\nu\to 0 \\ B_m{}^\nu\to 0 \end{matrix} \quad (m=2, 3, \cdots N_0)$$

Since $\{g_\nu\}$ has the three point property and since $\{J(g_\nu)\}$ is bounded, $\{g_\nu\}$ is equicontinuous on $[0, 2\pi]$. Thus it has a subsequence, call it $\{g_\nu\}$ again, which converges uniformly to a representation g.

Suppose h is the harmonic surface corresponding to g and h^ν, $\nu=0, 1, 2, \cdots$ are the harmonic surfaces corresponding to g_ν. Then h^ν converges uniformly to h and thus, by Harnack's theorem, the derivatives of h^ν converge uniformly to the derivatives of h in every closed subdomain of B.

Now if r_0, $0<r_0<1$, is any fixed number, then

$$r_0 h_r{}^\nu(r_0, \theta)h_\theta{}^\nu(r_0, \theta)=\sum_{m=2}^{\infty} r_0{}^m(A_m{}^\nu \cos m\theta+B_m{}^\nu \sin m\theta).$$

Note that $r_0^m A_m^\nu$ and $r_0^m B_m^\nu$ are the Fourier coefficients of the function

$$r_0 h_r^\nu(r_0, \theta) h_\theta^\nu(r_0, \theta), \quad \nu=0, 1, 2, \cdots$$

Since this sequence of functions converges uniformly to

$$r_0 h_r(r_0, \theta) h_\theta(r_0, \theta)$$

on ∂B_{r_0} as $\nu \to \infty$, it follows that $r_0^m A_m^\nu$ and $r_0^m B_m^\nu$ converge to the Fourier coefficien ts of the limit function. Thus if we write

$$r_0 h_r(r_0, \theta) h_\theta(r_0, \theta) = \sum_{m=2}^{\infty} r_0^m (A_m \cos m\theta + B_m \sin m\theta),$$

then

$$A_m = \lim_\nu A_m^\nu, \quad B_m = \lim_\nu B_m^\nu, \quad m=2, 3, \cdots\cdots$$

and hence $A_m = B_m = 0$, $m=2, 3, \cdots N_0$.

If we now rename the limit representation g as g_{N_0} emphasizing that it is obtained using N_0 terms and if we consider the sequence $\{g_N\}$ of representations, each g_N being obtained by using the same procedure as above and all from the same initial representation g_0, then it is clear that $\{g_N\}$ satisfies the three point condition by construction and $\{J(g_N)\}$ is bounded. Thus $\{g_N\}$ is equicontinuous on ∂B so that we may assume that this sequence converges uniformly to a representation g. And if h is the corresponding harmonic surface, then it is a minimal surface bounded by the given Jordan curve Γ. This shows that choosing N_0 sufficiently large from the begining we will end up with a surface which approximates a minimal surface very closely.

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