J. Korean Math. Soc.
Vol. 13, No. 1, 1976

A CHARACTERIZATION OF THE SIMPLE GROUPS $U_4(2)$ AND $L_4(2)$ BY A NON-CENTRAL INVOLUTION

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1. Introduction.

The four-dimensional projective unimodular unitary group $PSU_4(q^2)$ and the four-dimensional projective unimodular linear group $PSL_4(q)$, where $q=2^n>2$, have been characterized by the structure of the centralizer of a non-central involution in [3]. In this paper we will have the similar characterization for the case $q=2$. Denote $PSU_4(2^2)$ and $PSL_4(2)$ by $U_4(2)$ and $L_4(2)$, respectively. The characterization is contained in the following theorem:

THEOREM. *Let H_2 be the centralizer in $U_4(2)$ of a non-central involution of $U_4(2)$. If G is a finite group containing an involution z such that the centralizer H of z in G is isomorphic to H_2, then one of the following holds:*

(i) *G contains a normal subgroup of index 2,*

(ii) *$G=O_2(G)L$, where $O_2(G)$ is elementary abelian of order 2^4 and L is isomorphic to $L_2(4)$,*

(iii) $G\cong U_4(2)$, *or*

(iv) $G\cong L_4(2)$.

The method used in proving this theorem is essentially the same as that in [3]. The structure of the group $U_4(2)$ and the properties of the subgroup H_2 have been discussed in [3]. Thus some of the proofs in section 2 and section 4 will be omitted. The group $L_4(2)$ is isomorphic to the alternating group of degree 8, and the structure of this group has been studied in [4] and [6]. The author wishes to express his gratitude to Professor M. Suzuki for his guidance in the preparation of this paper.

2. The structure of $U_4(2)$.

Let F be a finite field with 4 elements. Then the group $U_4(2)$ has the following properties:

(2.1) Define

$$U_1=\{x_1(\alpha):\alpha\in F\},\quad x_1=x_1(1),$$

$$U_2=\langle x_2(1)\rangle,\quad x_2=x_2(1),$$

$$U_3=\{x_3(\gamma):\ \gamma\in F\},\quad x_3=x_3(1),$$

$$U_4=\langle x_4(1)\rangle,\quad x_4=x_4(1).$$

Then U_i is an elementary abelian subgroup whose multiplication is given by

$$x_i(\alpha)x_i(\beta)=x_i(\alpha+\beta).$$

Received by the editors Jan. 23, 1976.

(2.2) $U^1=U_1U_2U_3U_4$ is a Sylow 2-group of $U_4(2)$ of order 2^6, and $Z(U^1)=U_4$. Each element of U^1 is uniquely expressed as a product $x_1(\alpha)x_2(\beta)x_3(\gamma)x_4(\delta)$. We have

$$[x_1(\alpha),\ x_2]=x_3(\alpha)x_4,\quad \alpha\neq 0,$$

$$[x_1(\alpha), x_3(\gamma)]=x_4,\quad \alpha\neq\gamma \text{ and } \alpha\gamma\neq 0,$$

and all other types of commutators between elements of the various U_i are trivial.

(2.3) The subgroup $K^1=\{h(\mu):\ \mu\in F,\ \mu\neq 0\}$ is cyclic of order 3. It is a complement of U^1 in the normalzer B^1 of U^1, and the action of K^1 on B^1 is given by

$$h(\alpha)h(\beta)=h(\alpha\beta),\quad h(1)=1,$$

$$x_i(\alpha)^{h(\mu)}=x_i(\mu\alpha),\quad i=1,3,$$

$$[U_2U_4, K^1]=1.$$

(2.4) The subgroup $\langle n_1, n_2\rangle$ generated by two involutions n_1 and n_2 is a dihedral group of order 8, where $(n_1n_2)^4=1$. The subgroup K^1 is inverted by n_1 and centralized by n_2.

(2.5) The involutions n_1 and n_2 transform the elements of U^1 as follows:

$$n_1x_1(\alpha)n_1=x_1(\alpha^2)h(\alpha^2)n_1x_1(\alpha^2),$$

$$n_2x_2n_2=x_2n_2x_2,$$

$$n_1x_3(\gamma)n_1=x_3(\gamma^2),\quad n_1x_4n_1=x_2,$$

$$n_2x_3(\gamma)n_2=x_1(\gamma),\quad n_2x_4n_2=x_4.$$

In particular $(n_1x_1)^3=1$ and $(n_2x_2)^3=1$.

(2.6) The subgroups B^1 and $N^1=\langle K^1,\ n_1,\ n_2\rangle$ form a (B,N)-pair of $U_4(2)$.

3. Necessary lemmas.

The following lemmas will be used in this paper. Denote the dihedral group of order $2m$ by D_{2m}.

(3.1) Let u and v be two involutions of a finite group G such that u is not conjugate to v in G. Then there is an involution w in G such that:
(i) w commutes with both u and v, and
(ii) wu is conjugate in G to either u or v.
Moreover, if wu is conjugate to u [resp. v], then wv is conjugate to v [resp. u].

Proof. This is a well-known result. The proof may be found in [4].

(3.2) Let G be a finite group with a Sylow 2-group S of order 2 and let K be a complement of S in $N_G(S)$. If $|G:N_G(S)|=p$, p prime, then $G=NK$ with $N\cap K=1$

where N is a normal subgrop generated by all involutions and $N\cong D_{2p}$.
In particular if $p=3$ then $G=N\times K$.

Proof. Let u and v be two distinct involutions in G. Let $N=\langle u, v\rangle$. Then it is easy to see that $|uv|$ is odd and $C_N(u)=\langle u\rangle$. Since $C_G(S)=N_G(S)$ it follows that $|N|=2p$ and N contains all involutions of G. Now the Frattini argument yields the assertion. If $p=3$, there are exactly three involutions in G, and K centralizes N.

(3.3) Let G be a finite group with an elementary abelian Sylow 2-group $S=\langle u\rangle\times\langle v\rangle$ of order 4. Suppose that G is not 2-closed and S is the only Sylow 2-group of G containing the involution uv.

(i) If $N_G(S)\neq C_G(S)$, then G contains a normal subgroup isomorphic to $L_2(4)$.

(ii) If $N_G(S)=C_G(S)$, $|G:N_G(S)|=p^2$, p prime, and $S\cap S_1=1$ for some Sylow 2-group S_1, then G contains a normal subgroup isomorphic to $D_{2p}\times D_{2p}$.

Proof. Consider the case (i). The three involutions of S are conjugate, and G is a (TI)-group which is not 2-closed. Hence this assertion follows from (4.2) of [4].

Consider the case (ii). By Burnside lemma three involutions of S lie in distinct conjugate classes in G.

Let u_1 and v_1 be involutions of G which are conjugate to u and v, respectively. Since they are not conjugate in G, there is an involution w which commutes with both u_1 and v_1 by (3.1). Hence $\langle u_1, w\rangle$ and $\langle v_1, w\rangle$ are Sylow 2-groups, and w must be conjugate to uv. By the assumption this implies that $\langle u_1, v_1\rangle$ is a Sylow 2-group. In particular u_1 and v_1 commute.

These arguments and the assumption in (ii) yield that $|G:C_G(u)|=|C_G(u):N_G(S)|=p$ and all conjugates of v in G are contained in $C_G(u)$. By applying (3.2) for $C_G(u)/\langle u\rangle$, we can see that the subgroup N_1 generated by all conjugates of v is isomorphic to D_{2p}. Moreover, N_1 is normal in G and centralized by any involution conjugate to u. Similarly the subgroup N_2 generated by all conjugates of u is isomorphic to D_{2p} and it is normal in G. Hence $N=\langle N_1, N_2\rangle=N_1\times N_2$ and N is normal in G.

4. Nonsimple cases.

For the remainder of this paper G stands for a group satisfying the condition of the Theorem.

The element x_3 is a non-central involution of $U_4(2)$. Hence we may define H_2 to be the centralizer of x_3 in $U_4(2)$. Thus

$$H_2=\langle x_1\rangle U_2U_3U_4 \cup \langle x_1\rangle U_2U_3U_4n_1\langle x_1\rangle$$

and its order is $2^5\cdot 3$.

Since the center of H_2 is $\langle x_3\rangle$ we will identify the subgroup H of G with H_2, and in particular $H=C_G(x_3)$. The following subgroups of H will be fixed throughout this paper:

Subgroups	Orders
$Q=\langle x_1\rangle U_2U_3U_4$	2^5
$T=U_2U_3U_4$	2^4
$U=\langle x_1, x_3, x_4\rangle$	2^3
$V=\langle x_2, x_3, x_4\rangle$	2^3
$E=U_3U_4$	2^3

(4. 1) $H=C_G(x_3)$ and it is isomorphic to the centralizer in $L_4(2)$ of a non-central involution of $L_4(2)$.

(4. 2) T is the maximal normal 2-subgroup of H, and it has a complement $\langle x_1, n_1\rangle$ in H with $(n_1x_1)^3=1$.

(4. 3) A Sylow 2-group Q of H has the following properties:
(i) $Z(Q)=[Q,Q]=\langle x_3, x_4\rangle$ and $Q/Z(Q)$ is elementary abelian, and
(ii) $Q=C_G(Z(Q))=C_H(x_4)=N_H(Q)$ and $N_G(Q)=N_G(Z(Q))$.

(4. 4) T is the unique elementary abelian subgroup of order 2^4 in H. If T_1 is an elementary abelian subgroup of order 2^4 in G such that $T_1\cap T$ contains an element conjugate to x_3, then $T_1=T$.

(4. 5) Elementary abelian subgroups E and U_4 are characteristic in Q.
V is an elementary abelian normal subgroup of H which is the union of all conjugates of $Z(Q)$ in H.

(4. 6) There are five conjugate classes of involutions of T in H:
(i) x_3 (ii) $x_3x_4\sim x_2x_4\sim x_2x_3$
(iii) $x_4\sim x_2\sim x_2x_3x_4$ (iv) $x_2x_3(\gamma)x_4\sim x_2x_3(\gamma^2)x_4$
(v) $x_2x_3(\gamma)\sim x_2x_3(\gamma^2)\sim$all elements in $E-Z(Q)$
where γ is a fixed primitive cubic root of unity in F.

(4. 7) U and T are the only maximal elementary abelian subgroups of Q, where $U\cap T=Z(Q)$. No conjugate of T in G contains U.

(4. 8) There are exactly four involutions in $Q-T$, and any two of them are conjugate by an element of T. If w is an involution in $Q-T$, then $w\in U$ and $U=C_Q(w)$.

(4. 9) Define $S=N_G(Q)$. Then $|S|=2^5$ or 2^6.

Proof. By (4. 3) and (4. 5) the group S normalizes $Z(Q)$ and centralizes x_4. Hence either S centralizes $Z(Q)$ or cyclically permutes two elements in $Z(Q)-U_4$. Since $C_G(Z(Q))=Q$ by (4. 3), the assertion follows.

(4. 10) Assume that Q is a Sylow 2-group of G. Then G contains a normal subgroup of index 2.

Proof. It follows from (4. 9) that $N_G(Q)=Q$. By Grün [2] and (4. 3) this implies that the focal subgroup Q^* of Q in G is

$$Q^*=\langle Q\cap Z(Q)^x:\ x\in G\rangle.$$

By (4.5) we have $V\subseteqq Q^*$ and it suffices to show that $Q^*\subseteqq T$. Suppose that there is an element w in $Q-T$ which is conjugate to some involution z of $Z(Q)$. Then $w\in U-Z(Q)$ by (4.8), and $z^x=w$ and $U\subseteqq Q^x$ for some x in G. From (4.7) and (4.4) we can see that $U=U^x$ and $Z(Q)$ is not contained in T^x. This implies that two involutions in $Z(Q)-T^x$ are conjugate by (4.8). But no two involutions of $Z(Q)$ can be conjugate by Burnside lemma. This is a contradiction.

If Q is a Sylow 2-group of G, then (4.10) yields the case (i) of Theorem. For the remainder of this paper, therefore, we will assume that Q is not a Sylow 2-group of G. Note that $S=N_G(Q)$ is of order 2^6.

(4.11) $Z(S)=U_4$. Two involutions x_3 and x_3x_4 contained in $Z(Q)-U_4$ are conjugate in S, and x_3 and x_4 are not conjugate in G.

Proof. This follows from the proof of (4.9).

(4.12) T *is the unique elementary abelian subgroup of order* 2^4 *in* S.
In particular $N_G(S)\subseteqq N_G(T)$.

Proof. Suppose that S contains another elementary abelian subgroup T_1 of order 2^4. Then $|T_1\cap Q|\geqq 2^3$, and from (4.7) and (4.4) it is easy to see that $T_1\cap Q\subseteqq T$. Since all involutions in $Z(Q)-U_4$ are conjugate to x_3 by (4.11), it follows from (4.4) that $T_1\cap Z(Q)^h=U_4{}^h$ for all h in H. In particular T_1 contains x_2x_4 which is conjugate to x_3. This implies that $T_1=T$ by (4.4). But this is a contradiction.

(4.13) Let x be any element in $S-Q$. Then

$$T=E\cup V\cup V^x \quad \text{and} \quad E\cap V=E\cap V^x=V\cap V^x=Z(Q).$$

Proof. Since S normalizes E and $Z(Q)$ by (4.5), it suffices to show that $V^x\neq V$. Suppose that $V^x=V$. Then $x_2{}^x$ is contained in $V-Z(Q)$. Since $x_2{}^x$ is conjugate to x_4, from (4.6) and (4.11) it is easy to see that $x_2{}^x x_3$ is conjugate to x_3. The element x^2 is in Q, and it centralizes x_3. Hence $x_2x_3{}^x$ is conjuate to x_3. But $x_2x_3{}^x=x_2x_3x_4$ by (4.11), and it is conjugate to x_4 by (4.6). This is a contradiction.

(4.14) (i) Each element of $T-E$ is conjugate in S to either x_2 or x_2x_4.
(ii) Let A be a subgroup of T containing U_4. If A is not contained in E, then $A-E$ contains both an element conjugate to x_3 and an element conjugate to x_4.

Proof. This follows from (4.6), (4.11) and (4.13).

(4.15) We have $N_G(S)=N_G(T)\cap C_G(x_4)=N_G(E)$.

Proof. By (4.11) and (4.12) we have $N_G(S)\subseteqq N_G(T)\cap C_G(x_4)$. Let x be any element of $N_G(T)\cap C_G(x_4)$. Then it is easy to see from (4.14) and (4.11) that $Z(Q)^x\subseteqq E$. Since all elements in $E-Z(Q)$ are conjugate by (4.6), this implies that $E^X=E$

by (4.14) (ii). Hence we have $N_G(T) \cap C_G(x_4) \subseteqq N_G(E)$.

Let k be any element of $N_G(E)$. Then from (4.6) and (4.11) we can see that $U_4 \subseteqq Z(Q)^k \subseteqq E$. By the action of the involution x_1 on E this implies that Q normalizes $Z(Q)^k$, and $Q \subseteqq N_G(Q)^k = S^k$ by (4.3)(ii). Hence $k \in N_G(S)$. This proves the assertion.

(4.16) Each involution of T is conjugate in $N_G(T)$ to either x_3 or x_4. Moreover one of the following holds:

(i) $|N_G(T)| = 2^6 \cdot 3^2$ and $N_G(S) = N_G(Q) = S$. Conjugate classes of T listed in (4.6) are fused as follows: (i)$\sim$(ii)$\sim$(iv), (iii)$\sim$(v), or

(ii) $|N_G(T)| = 2^6 \cdot 3 \cdot 5$, $|N_G(S):S| = 3$ and $S = N_G(Q)$. Conjugate classes of T listed in (4.6) are fusedas follows: (i)$\sim$(ii)$\sim$(v), (iii)$\sim$(iv).

Proof. This follows from (4.11), (4.13) and (4.15). We remark that $|H| = |C_G(x_3)| = 2^5 \cdot 3$.

(4.17) Let v be any element in $T-E$. Then $C_G(v) \cap C_G(x_4)$ is 2-closed with Sylow 2-group T.

Proof. By (4.13) we can see that $C_S(v) = C_Q(v) = T$. Since S is the only Sylow 2-group of $C_G(x_4)$ which contains T by (4.15), it follows that T is a Sylow 2-group of $C_G(v) \cap C_G(x_4)$. Let T_1 be a Sylow 2-group of $C_G(v) \cap C_G(x_4)$. Then T_1 contains $\langle v, x_4 \rangle$, and it contains an element conjugate to x_3 by (4.14). Hence we have $T_1 = T$ by (4,4).

(4.18) Suppose that w is an involution in $S-T$. Let $P = \langle w \rangle T$ and $R = C_P(w)$. Then

(i) $C_S(w)$ is a non-abelian subgroup of order 2^4,

(ii) $R = \langle w \rangle Z(P)$ and it is the unique maximal elementary abelian subgroup of order 2^3 in P, and

(iii) $S = C_S(w)T$, $C_T(w) = Z(P) = [P, P] \subseteqq E$ and $P/Z(P)$ is elementary abelian of order 4.

Proof. If $w \in Q$ then the assertion follows from (4.8) and (4.3), where $R = U$. If (ii) of (4.16) holds, then S is the union of three conjugates of Q in $N_G(S)$, and the above fact yields the result.

Assume that (i) of (4.16) holds and there is an involution w in $S-Q$. Since $P = \langle w \rangle T$ and w is an involution, it is easy to see that $C_T(w) = Z(P)$ and $P/Z(P)$ is elementary abelian. By (4.11), (4.13) and (4.16) the action of w on T yields that $[P, P] = Z(P) \subseteqq E$ and $|Z(P)| = 4$. Since all involutions in $P-T$ are contained in $\langle w \rangle Z(P)$, the assertion follows.

(4.19) S is a Sylow 2-group of G.

Proof. Since S is a Sylow 2-group of $N_G(T)$ and $N_G(S) \subseteqq N_G(T)$, the assertion follows from the property of p-groups.

(4.20) If (i) of (4.16) holds, then $N_G(T)$ contains a normal subgroup of index 2.

Assume that (ii) of (4.16) holds. Then T is the maximal normal 2-subgroup of $N_G(T)$, and $N_G(T)$ contains a complement L of T such that $L \cong L_2(4)$.

Proof. If (i) of (4.16) holds, then $N_G(T)/T$ contains a self-normalizing abelian Sylow 2-group S/T, and $N_G(T)/T$ has a normal 2-complement by Burnside's Transfer theorem. This yields the assertion.

Assume that (ii) of (4.16) holds. Let w be an involution in $Q-T$. It is easy to see from (4.18) that $S=C_S(w)Z(Q)^k$ and $C_S(w) \cap Z(Q)^k = U_4$ for any k in $N_G(S)-S$. Hence $C_S(w) \cap U^k$ is of order 4, and there is some involution in $S-Q$ which commutes with w. This implies that S splits over T. By Gaschütz [1], the group $N_G(T)$ contains a complement L of T whose order is $2^2 \cdot 3 \cdot 5$. By (3.2) this yields the assertion.

5. Identification of G with $L_4(2)$

If T is normal in G, then (4.20) yields the cases (i) and (ii) of Theorem. In order to finish the proof of Theorem we will show in section 5 and section 6 that G is isomorphic to either $U_4(2)$ or $L_4(2)$ if T is not normal in G.

This section will be devoted to proving that G is isomorphic to $L_4(2)$ under the assumption that T is not normal in G and (i) of (4.16) holds. Note that in this case we have $S=N_G(S)=N_G(Q)$.

(5.1) $C_G(x_4)$ is not 2-closed.

Proof. Suppose that $C_G(x_4)$ is 2-closed. Then $C_G(x_4)=S$ by (4.15), and the following arguments yield the contradiction:

(i) $T^x \cap T = T$ or 1 for any x in G.

If $T^x \cap T$ contains an element conjugate to x_3, then $T^x=T$ by (4.4). If $T^x \cap T$ contains an element conjugate to x_4, then both T^x and T are contained in some conjugate of S, which yields that $T^X=T$ by (4.12).

(ii) No element of $S-T$ is conjugate to an element of T.

Let w be an element in $S-T$. If w is not an involution, then w is not conjugate to any element of T. Assume that w is an involution. To prove (ii) it suffices to show that $C_S(w)$ is a Sylow 2-group of $C_G(w)$ by (4.18)(i) and (4.16). Suppose not. Since S is the only Sylow 2-group of $N_G(T)$ containing $C_G(w)$ by (4.18)(iii), it follows that there is an element x in $G-N_G(T)$ such that $C_G(w) \cap S^x$ is a Sylow 2-group of $C_G(w)$ containing $C_S(w)$. Then $T^x \cap T=1$ by (i) and it is easy to see that $C_S(w)$ is elementary abelian. But this contradicts (4.18).

(iii) Take any element x in $G-N_G(T)$. Then $x_3{}^x$ and x_4 are not conjugate in G, and there is an involution w which commutes with both $x_3{}^x$ and x_4 by (3.1). Hence $w \in H^x \cap S$, and it follows from (i) and (ii) that $w \in S-T$. This implies that wx_4 is in $S-T$ and it is not conjugate to any element of T. But this contradicts (3.1).

(5.2) For any element x in $C_G(x_4)-S$ the following holds:

(i) $S^x \cap T=E$, $E^x \cap E = T^x \cap T = U_4$, $U^x=U$, $Q^x \cap E = Z(Q)$, and

(ii) $S \cap T^x = E^x$, $Q \cap E^x = Z(Q)^x$, $S=TE^x$.

Proof. Since $T^x \neq T$ by (4.15), it follows from (4.14) and (4.4) that $U_4 \subseteqq T^x \cap T = E^x \cap E \neq E$. Suppose that $S^x \cap T$ is not contained in E. Then $S^X - T^X$ contains an involution w in $T-E$, which implies that $C_G(w) \cap C_G(x_4)$ is 2-closed with Sylow 2-group T by (4.17). But this contradicts (4.18). Hence $S^x \cap T \subseteq E$.

The involutions $x_3{}^x$ and x_4 are in $C_G(x_4)$ and they are not conjugate in $C_G(x_4)$. Hence there is an involution w in $H^x \cap C_G(x_2) \cap C_G(x_4)$ satisfying (3.1). Since Q^x is a normal Sylow 2-group of $H^x \cap C_G(x_4)$ by (4.3) (ii), it follows from (4.17) that $w \in Q^x \cap T \subseteq E$. The similar argument yields that wx_2 of $T-E$ can not be conjugate in $C_G(x_4)$ to x_3. Hence wx_2 is conjugate to x_2, and $wx_3{}^x$ is conjugate to $x_3{}^x$ in $C_G(x_4)$. Clearly no element of $E^x \cap E$ is conjugate to x_3. By considering conjugate classes in (4.16) we can show that $w \notin E^x \cap E$. This implies that $w \in Q^x - T^x$, and w is conjugate to $wx_3{}^x$. Hence w is conjugate to x_3, and it is contained in $Z(Q) - U_4$. Thus we have $Q^x \cap T = Z(Q)$ and $T^x \cap T = U_4$. Since $Q = C_G(Z(Q))$ by (4.3), by applying (4.18) for S^x we obtain that $S^x \cap Q$ is of order 2^4 and $U = Q^x \cap Q = U^x$. Now it is easy to show the assertion (i).

Since the assertion (i) holds for any x in $C_G(x_4) - S$, the assertion (ii) follows.

(5.3) S contains exactly three normal subgroups Q, P_1 and P_2 of index 2 which contain T. Each P_i contains a unique maximal elementary abelian subgroup R_i of order 2^3. Each R_i is normal in $C_G(x_4)$ and $R_i \cap T = Z(P_i)$.

Proof. By (5.2) the group S contains a complement of T which is a conjugate of U_3 in $C_G(x_4)$. In particular $S-Q$ contains involutions. Hence, by (4.18), it suffices to show that R_i is normal in $C_G(x_4)$.

Let x be any element of $C_G(x_4) - S$. By (5.2) and (4.18) it is easy to see that the centralizer of $P_1 \cap E^x$ in $C_G(x_4)$ is a subgroup of order 2^4 in S and the centralizer of $P_1 \cap E^x$ in S^x is either $P_1{}^x$ or $P_2{}^x$. Again by (4.18) this yields that $R_1 = P_1 \cap P_1{}^x = R_1{}^x$ or $R_1 = P_1 \cap P_2{}^x = R_2{}^x$. Since $C_G(x^4)/S$ is of odd order and R_i is normalized by S, it follows that $R_1 = R_1{}^x$.

(5.4) Define D to be the intersection of two distinct Sylow 2-groups of $C_G(x_4)$. Then

(i) All conjugates of E in $C_G(x_4)$ are contained in D,

(ii) D is the maximal normal 2-subgroup of $C_G(x_4)$ and D/U_4 is elementary abelian, and

(iii) U, R_1 and R_2 are contained in D.

Proof. This follows from (5.2) and (5.3).

(5.5) If x is an element of S such that $x^2 \in U_4$, then either $x \in T$ or $x \in D$.

Proof. By (5.3) and (4.13) it is easy to see that x is contained in either T or $EU \cap ER_1 \cap ER_2$. Hence the assertion follows from (5.4).

(5.6) $|C_G(x_4):S| = 3$. Moreover, $C_G(x_4)$ contains a subgroup M which satisfies the following:

(i) M is a complement of D which is a dihedral group of order 6, and

(ii) $S \cap M$ is either $\langle x_2 \rangle$ or $\langle x_2x_4 \rangle$.

Proof. From (5. 3) and (5. 2) it follows that there are 12 involutions in $S-T$, and each conjugate of E, $\neq E$, in $C_G(x_4)$ contains exactly 6 involutions in $S-T$. This yields that $|C_G(x_4):N_G(E)|=3$. Since $S=N_G(S)=N_G(E)$ by (4. 15), we have $|C_G(x_4):S|=3$, and $C_G(x_4)/D$ is a dihedral group of order 6 by (3. 2).

By (5. 4) the group $C_G(x_4)/U_4$ contains an abelian normal subgroup D/U_4, and S/U_4 contains a complement U_2U_4/U_4 of D/U_4. Hence, by Gaschütz, there is a subgroup M_1 such that $C_G(x_4)=DM_1$ and $D\cap M_1=U_4$. Since $S\cap M_1/U_4$ is of order 2, it follows from (5. 5) that $S\cap M_1 \subseteq T$, and $S\cap M_1$ splits over U_4. Again by Gaschütz this implies that M_1 contains a complement M of U_4. Hence M is a complement of D in $C_G(x_4)$. Since $S\cap T$ is contained in T but not in E, it follows from (4. 14) (i) that we may assume that M itself satisfies the condition.

(5. 7) There is an involution s in $C_G(x_4)$ which satisfies the following:

(i) $(sx_2)^3=1$ and $C_G(x_4)=\langle D, s, x_2\rangle$, and

(ii) $U_3{}^s$ is a complement of T in S such that $x_3{}^s=x_1$.

Proof. M is generated by two involutions e and f such that $S\cap M=\langle e\rangle$ and $(ef)^3=1$. Since either $e=x_2$ or $e=x_2x_4$, either $s=f$ or $s=fx_4$ satisfies (i). By (5. 2) we have $x_3{}^s\in Q-T$. Since any conjugate of s by an element of T satisfies (i), it follows from (4. 8) that we may assume that s itself satisfies (ii).

(5. 8) Let u and v be involutions of U_3. Then

(i) $[u^s, x_2]=x_3x_4$ or u according as $u=x_3$ or $u\neq x_3$, and

(ii) $[u^s, v]=1$ or x_4 according as $u=v$ or $u\neq v$.

Proof. If $u=x_3$ then the assertion follows from (5. 7) and (2. 2). By (5. 3) the involution s normalizes R_1 and R_2. Hence the assertion can be easily seen by using (4. 18) and (4. 16).

(5. 9) The group G is isomorphic to $L_4(2)$.

Proof. The structure of $C_G(x_4)$ is determined by (5. 7) and (5. 8). Let ω be a fixed primitive cubic root of unity in F. Then each element γ of F is uniquely expressed as $\gamma=\xi+\eta\omega$, where ξ and η are either 0 or 1. Define a map from $C_G(x_4)$ into $L_4(2)$ which sends

$$x_3(\gamma)^s \quad \text{into} \quad \begin{pmatrix} 1 & & & \\ \xi & 1 & & \\ & & 1 & \\ & & \xi+\eta & 1 \end{pmatrix},$$

$$x_2(\beta)x_3(\gamma)x_4(\delta) \quad \text{into} \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ \xi & \beta & 1 & \\ \delta & \xi+\eta & & 1 \end{pmatrix}, \quad \gamma=\xi+\eta\omega,$$

$$s \quad \text{into} \quad \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}.$$

Then it is easy to see that this map defines an isomorphism of $C_G(x_4)$ onto the centralizer in $L_4(2)$ of a central involution of $L_4(2)$. Hence G is isomorphic to $L_4(2)$ by Theorem of [6].

We remark that (5.9) can be proved by constructing a (B, N)-pair of G as in [4]. The structure of $N_G(T)$ can be determined by using (3.3) and that of $N_G(R_i)$ can be determined by using (5.3) and the argument in [4].

6. Identification of G with $U_4(2)$

This section will be devoted to proving that G is isomorphic to $U_4(2)$ under the assumption that T is not normal in G and (ii) of (4.16) holds. Note that $|N_G(S):S|=3$.

(6.1) $C_G(x_4)$ is not 2-closed. For any element x in $C_G(x_4)-N_G(S)$ the following holds:

(i) $S^x\cap T=E$, $S\cap T^x=E^x$, $E^x\cap E=T^x\cap T=U_4$, and

(ii) $S=TE^x$ and $U^x=U^k$ for some k in $N_G(S)$.

Proof. The proof is similar to those of (5.1) and (5.2). Note that S is the union of three conjugates of Q in $N_G(S)$.

(6.2) Define D to be the intersection of two distinct Sylow 2-groups of $C_G(x_4)$. Then

(i) All conjugates of E and all conjugates of U in $C_G(x_4)$ are contained in D, and

(ii) D is the maximal normal 2-subgroup of $C_G(x_4)$, and D/U_4 is elementary abelian.

Proof. This follows from (6.1).

(6.3) Each involution of G is conjugate to either x_3 or x_4. Moreover,

(i) Each involution in $S-T$ is conjugate to x_3 in $C_G(x_4)$, and

(ii) $C_S(x_2)=T$ and an element of S conjugate to x_4 is either x_4 or a conjugate of x_2 in S.

Proof. The assertion can be easily seen from (4.16). Since S is the union of three conjugates of Q in $N_G(S)$, it follows from (4.8) that all involutions in $S-T$ are conjugate in $N_G(S)$. On the other hand $S-T$ contains a conjugate of x_3 in $C_G(x_4)$ by (6.1). This proves the assertion.

(6.4) $N_G(T)$ contains subgroups L and K which satisfy the following:

(i) L is a complement of T in $N_G(T)$ which is isomorphic to $L_2(4)$,

(ii) $H\cap L=\langle x_1, n_1\rangle$ with $(n_1x_1)^3=1$, and

(iii) K is a complement of S in $N_G(S)$ which is inverted by n_1 and contained in L.

Proof. By (4.20) there is a subgroup L satisfying (i). The abelian normal subgroup T of H has a complement in Q. Moreover, all complement of T in Q are conjugate in Q by (4.8). By Gaschütz it follows from (4.2) that all complements of T in H are conjugate to $\langle x_1, n_1\rangle$ in H. Hence we may assume that L satisfies both (i) and (ii). By the property of $L_2(4)$ it is easy to see the existence of the required subgroup K.

(6.5) Define $W=S\cap L$. Then

(i) K acts regularly on $U_3^\#$ and $W^\#$, and $C_T(K)=U_2U_4$,
(ii) K normalizes exactly two Sylow 2-groups S and S^{n_1}, and
(iii) $N_G(T)=\langle T, K, n_1, x_1\rangle$.

Proof. $K\subseteq N_G(E)\subseteq C_G(x_4)$ by (4.15). Moreover, $U_3=E\cap E^{n_1}$ and $U_2=U_4^{n_1}$. By (6.4)(iii) this yields that K normalizes U_3 and centralizes U_2U_4. Since $K\cap C_G(x_3)=1$, it follows that K acts regularly on $U_3^\#$ and we have $C_T(K)=U_2U_4$. The remaining assertions follow from the property of $L_2(4)$.

(6.6) The statement in (5.5) holds.

Proof. Let x be an element of Q such that $x^2\in U_4$. By the structure of Q it follows that either $x\in T$ or $x\in EU\subseteq D$. Hence the assertion follows from (6.2) since S is the union of conjugates of Q in $N_G(S)$.

(6.7) $|C_G(x_4):N_G(S)|=3$. Moreover, $C_G(x_4)$ contains a complement M of D which is the direct product of K and a dihedral group of order 6.

Proof. As in (5.6), by using (6.1) and (4.15) we can show that $|C_G(x_4):N_G(S)|=3$. Hence, by (3.2), $C_G(x_4)/D$ is the direct product of K and a dihedral group of order 6. Again by using (6.2) and (6.6) we can repeat the same argument as that in (5.6) to show that $C_G(x_4)$ splits over D. This proves the assertion.

(6.8) There is an involution s in $C_G(x_4)$ which satisfies the following:
(i) $(sx_2)^3=1$ and $C_G(x_4)=\langle D, K, s, x_2\rangle$,
(ii) s centralizes K, and
(iii) $W=U_3^s$ and $x_1=x_3^s$.

Proof. Since $S\cap M$ is contained T and is centralized by K, it follows from (6.5)(i) that either $S\cap M=\langle x_2\rangle$ or $S\cap M=\langle x_2x_4\rangle$. Note that K is a direct factor of M by (6.7). Now the similar argument to that in (5.7) yields the existence of the involution s satisfying (i) and (ii). Since any involution of $S-T$ is contained in either E^s or $E^{x_2sx_2}$, and x_2sx_2 also satisfies (i) and (ii), it follows from (6.5)(i) that we may assume that $W\subseteq E^s$. It is easy to see that U_3 is the only subgroup of order 4 in E which is normalized by K. Hence we have $W^s=U_3$. By (6.1)(ii) it can be easily seen that $U^s=U$. Thus $\langle x_1\rangle=W\cap U=(U_3\cap U)^s=\langle x_3\rangle^s$. This proves the assertion.

(6.9) The following holds:
(i) Define $k(\gamma)$ to be an element of K such that $x_3^{k(\gamma)}=x_3(\gamma)$. Then $x_1^{k(\gamma)}=x_3(\gamma)^s$.
(ii) Let u and v be involutions in U_3. Then $[u^s, x_2]=ux_4$, and $[u^s, v]=1$ or x_4 according as $u=v$ or $u\neq v$.

Proof. The assertion (i) follows from (6.5) (i) and (6.8)(iii). By (2.2), (6.5) and (6.8) the assertion (i) yields (ii).

(6.10) Let v be any conjugate of x_4 which is not in T. Then v centralizes the unique conujgate of x_4 which lies in T.

Provf. Since v and x_3 are not conjugate in G, there is an involution w in $C_G(v)\cap H$

by (3.1). By (6.3) no element of $H-T$ is conjugate to x_4. Hence if w is conjugate to x_4, then w is in T, and w is the required conjugate. Assume that w is conjugate to x_3. Since $w\in Q^h$ for some $h\in H$, it follows from (6.3) that $w\in T^{xh}$ for some $x\in C_G(x_4)$. Since $C_G(w)$ is conjugate to H, the above argument yields that $v\in T^{xh}$, and v centralizes $x_4{}^{xh}$ which lies in T.

To show the uniqueness suppose that v centralizes two distinct conjugates e and f of x_4 which lie in T. By using (4.16) and (4.17) we can easily see that $C_G(e)\cap C_G(f)$ is 2-closed with Sylow 2-group T. This implies that v is contained in T, which is a contradiction.

(6.11) Denote n_1 by t. Then $\langle t, s\rangle$ is a dihedral group of order 8 where $(ts)^4=1$.

Proof. By (6.5)(i) the involutions x_4 and x_2 are the only conjugates of x_4 contained in T which are centralized by K. Moreover $x_2=x_4{}^t$.

Let x be any element of $N_G(K)$. Then $x_4{}^x$ is centralized by K. Hence if $x_4{}^x\in T$, then either $x_4{}^x=x_4$ or $x_4{}^x=x_2$. Assume that $x_4{}^x\notin T$. By (6.10) there is a unique conjugate v of x_4 contained in T which is centralized by $x_4{}^x$. By uniqueness v is centralized by K, and either $v=x_4$ or $v=x_2$. If $v=x_4$, then $x_4{}^x$ is constained in either T^s or T^{sx_2}, and either $x_4{}^x=x_2{}^s$ or $x_4{}^x=x_2{}^{sx_2}=s$. Note that $(sx_2)^3=1$. Hence if $v=x_2$, then either $x_4{}^x=x_2{}^{st}$ or $x_4{}^x=s^t$. On the other hand, it follows from (6.3)(ii) and (6.5)(i) that $C_G(x_2)\cap C_G(x_4)=TK$.

From the above argument it follows that the conjugation defines a permutation representation of $N_G(K)$ on six letters whose kernel is K. Suppose that s transforms $x_2{}^{st}$ into s^t. Then an easy computation yields that $(sx_2)^3$ transforms $x_2{}^{st}$ into s^t, which contradicts (6.8). Hence s transforms $x_2{}^{st}$ into $x_2{}^{st}$, and we can easily see that $(ts)^4\in K$. Since s centralizes K and inverts $(ts)^4$ it follows that $(ts)^4=1$. Now the assertion is proved.

(6.12) The group G is the union of the following eight cosets:

$$B,\ BsU_2,\ BtW,\ BstU_4W,\ BtsU_2U_3,\ BstsT,\ BtstD,\ BststS,$$

where $B=N_G(S)$.

Proof. Let x be any element of G. By using (6.10) and (6.3)(ii) we consider the possibilities for $x_4{}^x$. The details are in [3].

(6.13) Each of the double cosets of (6.12) is of the form BdS_d, where $d\in\langle t, s\rangle$ and S_d is a complement of $S\cap S^d$ in S.

Proof. The proof is the same as that in [3].

(6.14) Each element of G is expressible in just one way in the form $x=bdy$ where $b\in B$, $d\in\langle t, s\rangle$ and $y\in S_d$.

Proof. We can show that the eight double cosets of (6.12) decompose G as the union of the double cosets with respect to B. This yields the assertion. For details see [3].

(6. 15) The group G is isomorphic to $U_4(2)$.

Proof. Define a map from $U_4(2)$ into G which sends

$$\begin{array}{lll} x_1(\alpha)x_2(\beta)x_3(\gamma)x_4(\delta) & \text{into} & x_3(\alpha)^s x_2(\beta)x_3(\gamma)x_4(\delta) \\ h(\mu) & \text{into} & k(\mu), \\ n_1 & \text{into} & n_1, \text{ and} \\ n_2 & \text{into} & s. \end{array}$$

By (6. 8), (6. 9), (6. 11), (6. 14) and (2. 1)—(2. 6) it is easy to see that this map is a homomorphism of $U_4(2)$ onto G. Since $U_4(2)$ is simple, this map is an isomorphism of $U_4(2)$ onto G.

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