J. Korean Math. Soc.
Vol. 13, No. 1, 1976

ON Q-KILLING TENSORS IN A QUATERNION KAEHLERIAN MANIFOLD

BY SANG-SEUP EUM

§ 0. Introduction.

In an n-dimensional Riemannian manifold, a vector field v_a is called a Killing vector if it satisfies the Killing equation:

$$\nabla_a v_b + \nabla_b v_a = 0,$$

where ∇ denotes the operator of the covariant differentiation with respect to the Riemannian connection.

A Killing tensor v_{cd} is, by definition, a skew symmetric tensor satisfying Killing–Yano's equation:

$$\nabla_b v_{cd} + \nabla_c v_{bd} = 0.$$

S. Tachibana [5] defined a conformal Killing tensor u_{cd} in a Riemannian manifold by the condition:

$$\nabla_b u_{cd} + \nabla_c u_{bd} = 2\rho_d g_{bc} - \rho_c g_{bd} - \rho_b g_{cd},$$

where ρ_c is a vector field and g_{bc} the Riemannian metric.

As a generalization of a Killing tensor in a Riemannian manifold, C-H. Chen [1] defined an F-Killing tensor w_{ba} in a Kaehlerian manifold $M(F, g)$ by the condition:

$$\nabla_b w_{cd} + \nabla_c w_{bd} = (F_b{}^r F_{cd} + F_c{}^r F_{bd}) \nabla^e w_{er}.$$

In the present paper we shall define a Q-Killing tensor in a quaternion Kaehlerian manifold as a generalization of the Killing tensor in a Riemannian manifold and the F-Killing tensor in a Kaehlerian manifold, and generalize some results on Killing tensors or conformal Killing tensors in a Riemannian manifold and F-Killing tensors in a Kaehlerian manifold to those on Q-Killing tensors in a quaternion Kaehlerian manifold.

§ 1. Preliminaries.

Let M be an almost quaternion manifold, that is, a $4n$-dimensional differentiable manifold which admits a set of three tensor fields F, G, H of type $(1,1)$ satisfying

$$\begin{gathered} F^2 = -I, \quad G^2 = -I, \quad H^2 = -I, \\ F = GH = -HG, \quad G = HF = -FH, \quad H = FG = -GF, \end{gathered} \tag{1.1}$$

I denoting the identity tensor.

Received by the editors Jan. 20, 1976.

In a previous paper [7], we proved that there exists a Hermitian metric g for the almost quaternion structure F, G, H, that is, a Riemannian metric satisfying

$$
\begin{aligned}
&g(FX,\ FY)=g(X,\ Y),\\
(1.2)\qquad &g(GX,\ GY)=g(X,\ Y),\\
&g(HX,\ HY)=g(X,\ Y)
\end{aligned}
$$

for arbitrary vector fields X and Y of M. In this case M is called an almost quaternion metric manifold.

If an almost quaternion metric manifold M satisfies the condition

$$
\begin{aligned}
&\nabla_X F=\ r(X)G-q(X)H,\\
(1.3)\qquad &\nabla_X G=-r(X)F+p(X)H,\\
&\nabla_X H=\ q(X)F-p(X)G,
\end{aligned}
$$

where ∇ is the operator of covariant differentiation with respect to g, p, q, r certain 1-forms and X an arbitrary vector field of M, then M is called a quaternion Kaehlerian manifold [4].

Let $\{U;\ x^h\}$ be a system of coordinate neighborhoods of M, where, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 4n\}$.

Since a quaternion Kaehlerian manifold is an Einstein space [4], the Ricci tensor has components of the form:

$$(1.4)\qquad K_{ji}=4k(n+2)g_{ji},$$

where $4n=\dim. M$. In this case, using the Ricci formula, we obtain from (1.3)

$$
\begin{aligned}
&K_{kjt}{}^hF_i^t-K_{kji}{}^tF_t^h=4k(G_{kj}H_i^h-H_{kj}G_i^h),\\
(1.5)\qquad &K_{kjt}{}^hG_i^t-K_{kji}{}^tG_t^h=4k(H_{kj}F_i^h-F_{kj}H_i^h),\\
&K_{kjt}{}^hH_i^t-K_{kji}{}^tH_t^h=4k(F_{kj}G_i^h-G_{kj}F_i^h),
\end{aligned}
$$

where $K_{kjt}{}^h$ is the curvature tensor of M.

In a quaternion Kaehlerian manifold M, the following formulas are known [4].

$$
\begin{aligned}
(1.6)\qquad &K_{kjts}F^{ts}=-8knF_{kj},\\
&K_{kjts}G^{ts}=-8knG_{kj},\quad K_{kjts}H^{ts}=-8knH_{kj},
\end{aligned}
$$

$$
\begin{aligned}
(1.7)\qquad &K_{ktsj}F^{ts}=4knF_{kj},\\
&K_{ktsj}G^{ts}=4knG_{kj},\quad K_{ktsj}H^{ts}=4knH_{kj}.
\end{aligned}
$$

Moreover, S. Ishihara proved the following [4]

THEOREM A. *A quaternion Kaehlerian manifold M ($\dim. M\geqq 8$) is of constant Q-sectional curvature $4k$ if and only if its curvature tensor has components of the form*

$$K_{kjih}=k(g_{kh}g_{ji}-g_{ki}g_{jh}+Q_{khji}-Q_{kijh}-2Q_{kjih}), \tag{1.8}$$

where the tensor Q_{khji} *is defined by*

$$Q_{khji}=F_{kh}F_{ji}+G_{kh}G_{ji}+H_{kh}H_{ji}. \tag{1.9}$$

§ 2. *Q*-Killing tensors.

In a $4n$-dimensional quaternion Kaehlerian manifold M ($n\geqq 2$), we shall call a skew symmetric tensor u_{jk} a *Q-Killing tensor* if it satisfies the condition

$$\nabla_j u_{kl}+\nabla_k u_{jl}=\frac{1}{3}(Q_j{}^t{}_{kl}+Q_k{}^t{}_{jl})\nabla^r u_{rt} \tag{2.1}$$

where the tensor $Q_j{}^t{}_{kl}$ is defined by (1.9), that is,

$$Q_j{}^t{}_{kl}=F_j{}^tF_{kl}+G_j{}^tG_{kl}+H_j{}^tH_{kl}.$$

We call $\nabla^r u_{rt}$ the *associated vector* of u_{ij}. If the associated vector of u_{ij} vanishes identically, then u_{ij} is a Killing tensor.

Operating ∇_i to (2.1) we get

$$\nabla_i\nabla_j u_{kl}+\nabla_i\nabla_k u_{jl}=\frac{1}{3}(Q_j{}^t{}_{kl}+Q_k{}^t{}_{jl})\rho_{it}, \tag{2.2}$$

where we have put

$$\rho_{it}=\nabla_i\nabla^r u_{rt}. \tag{2.3}$$

By interchanging indices i, j, k as $i\to j\to k\to i$ in this equation we obtain the following two equations:

$$\nabla_j\nabla_k u_{il}+\nabla_j\nabla_i u_{kl}=\frac{1}{3}(Q_k{}^t{}_{il}+Q_i{}^t{}_{kl})\rho_{jt}, \tag{2.4}$$

$$\nabla_k\nabla_i u_{jl}+\nabla_k\nabla_j u_{il}=\frac{1}{3}(Q_i{}^t{}_{jl}+Q_j{}^t{}_{il})\rho_{kt}. \tag{2.5}$$

If we form $(2.2)+(2.4)-(2.5)$, then it follows that

$$\begin{aligned}&2\nabla_i\nabla_j u_{kl}-2K_{kji}{}^t u_{lt}-K_{jkl}{}^t u_{it}-K_{ikl}{}^t u_{jt}-K_{jil}{}^t u_{kt}\\ &=\frac{1}{3}[(Q_j{}^t{}_{kl}+Q_k{}^t{}_{jl})\rho_{it}+(Q_k{}^t{}_{il}+Q_i{}^t{}_{kl})\rho_{jt}-(Q_i{}^t{}_{jl}+Q_j{}^t{}_{il})\rho_{kt}].\end{aligned} \tag{2.6}$$

Transvecting (2.6) with g^{kl} and making use of

$$(K_{jrit}+K_{irjt})u^{rt}=0, \tag{2.7}$$

we obtain

$$3(\rho_{ij}+\rho_{ji})+(Q_{itjr}+Q_{jtir})\rho^{rt}=0. \tag{2.8}$$

Transvecting (2.8) with $Q_k{}^i{}_l{}^j$, we find

$$3(\rho_{kl}+\rho_{lk})+5(Q_{ktlr}+Q_{ltkr})\rho^{rt}=0. \tag{2.9}$$

Comparing (2.8) with (2.9), we get

$$\rho_{ij}+\rho_{ji}=0, \qquad (Q_{itjr}+Q_{jtir})\rho^{rt}=0.$$

Thus we have the following

THEOREM 1. *In a quaternion Kaehlerian manifold, the associated vector of a Q-Killing tensor is a Killing vector.*

§ 3. *Q*-Killing tensors. (continued)

One of the purposes of the present paper is to prove the following

THEOREM 2. *In a quaternion Kaehlerian manifold, if there exists (locally) a Q-Killing tentor u_{ij} satisfying $u_{ij}(p)=c_{ij}$ for any point p and any constant c_{ij} $(=-c_{ji})$, then the manifold is of constant Q-sectional curvature.*

Proof. we shall deform (2.6) into another form. By a change of indices $j\to k\to l\to j$ in (2.6), we have

$$\begin{aligned}&2\nabla_i\nabla_k u_{lj}-2K_{lki}{}^t u_{jt}-K_{klj}{}^t u_{it}-K_{ilj}{}^t u_{kt}-K_{kji}{}^t u_{lt}\\ &=\frac{1}{3}[(Q_k{}^t{}_{lj}+Q_l{}^t{}_{kj})\rho_{it}+(Q_l{}^t{}_{ij}+Q_i{}^t{}_{lj})\rho_{kt}-(Q_i{}^t{}_{kj}+Q_k{}^t{}_{ij})\rho_{lt}],\end{aligned} \tag{3.1}$$

$$\begin{aligned}&2\nabla_i\nabla_l u_{jk}-2K_{jli}{}^t u_{kt}-K_{ljk}{}^t u_{it}-K_{ijk}{}^t u_{lt}-K_{lik}{}^t u_{jt}\\ &=\frac{1}{3}[(Q_l{}^t{}_{jk}+Q_j{}^t{}_{lk})\rho_{it}+(Q_j{}^t{}_{ik}+Q_i{}^t{}_{jk})\rho_{lt}-(Q_i{}^t{}_{lk}+Q_l{}^t{}_{ik})\rho_{jt}].\end{aligned} \tag{3.2}$$

Adding (2.6), (3.1) and (3.2) and taking account of (2.1), we get

$$\begin{aligned}&6\nabla_i\nabla_j u_{kl}-3(K_{lki}{}^t u_{jt}+K_{jli}{}^t u_{kt}+K_{kji}{}^t u_{lt})\\ &=\frac{1}{3}[(Q_k{}^t{}_{jl}-Q_l{}^t{}_{jk}-2Q_j{}^t{}_{lk})\rho_{it}+(Q_k{}^t{}_{il}-Q_l{}^t{}_{ik}-2Q_i{}^t{}_{lk})\rho_{jt}\\ &\quad+(Q_l{}^t{}_{ij}-Q_j{}^t{}_{il}-2Q_i{}^t{}_{jl})\rho_{kt}+(Q_j{}^t{}_{ik}-Q_k{}^t{}_{ij}-2Q_i{}^t{}_{kj})\rho_{lt}].\end{aligned} \tag{3.3}$$

By forming $(3.3)-3\times(2.6)$, we obtain

$$\begin{aligned}&3(K_{jkl}{}^t u_{it}+K_{ilk}{}^t u_{jt}+K_{lij}{}^t u_{kt}+K_{kji}{}^t u_{lt})\\ &=\frac{1}{3}[(Q_{lki}{}^t\rho_{jt}-Q_{lkj}{}^t\rho_{it})+(Q_{jli}{}^t\rho_{kt}-Q_{jlk}{}^t\rho_{it})\\ &\quad+(Q_{ikj}{}^t\rho_{lt}-Q_{ikl}{}^t\rho_{jt})+(Q_{ijl}{}^t\rho_{kt}-Q_{ijk}{}^t\rho_{lt})\\ &\quad+2(Q_{ilj}{}^t\rho_{kt}-Q_{ilk}{}^t\rho_{jt})+2(Q_{jki}{}^t\rho_{lt}-Q_{jkl}{}^t\rho_{it})].\end{aligned} \tag{3.4}$$

Next, we shall obtain algebraic relation between components of u_{ij} and ρ_{ij}.

Transvecting (3.4) with F^{jk} and taking account of (1.4)—(1.7) and the skew symmetric property of the tensor ρ_{ij}, we obtain after some computations

$$
\begin{aligned}
F_l{}^t\rho_{it}-F_i{}^t\rho_{lt}&=9k(F_l{}^tu_{it}-F_i{}^tu_{lt})+\frac{9}{4n}K_{li}{}^{rk}F_k{}^tu_{rt}\\
&\quad+\frac{9k}{n}(\lambda_3G_{li}-\lambda_2H_{li})+\frac{1}{2n}(\mu_3G_{li}-\mu_2H_{li}),
\end{aligned}
\tag{3.5}
$$

where we have put

$$
\lambda_1=F_k{}^tu_t{}^k,\quad \lambda_2=G_k{}^tu_t{}^k,\quad \lambda_3=H_k{}^tu_t{}^k,
$$

$$
\mu_1=F_k{}^t\rho_t{}^k,\quad \mu_2=G_k{}^t\rho_t{}^k,\quad \mu_3=H_k{}^t\rho_t{}^k.
$$

Similarly, transvecting (3.4) with G^{jk} or H^{jk} respectively, we obtain

$$
\begin{aligned}
G_l{}^t\rho_{it}-G_i{}^t\rho_{lt}&=9k(G_l{}^tu_{it}-G_i{}^tu_{lt})+\frac{9}{4n}K_{li}{}^{rk}G_k{}^tu_{rt}\\
&\quad+\frac{9k}{n}(\lambda_1H_{li}-\lambda_3F_{li})+\frac{1}{2n}(\mu_1H_{li}-\mu_3F_{li}),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
H_l{}^t\rho_{it}-H_i{}^t\rho_{lt}&=9k(H_l{}^tu_{it}-H_i{}^tu_{lt})+\frac{9}{4n}K_{li}{}^{rk}H_k{}^tu_{rt}\\
&\quad+\frac{9k}{n}(\lambda_2F_{li}-\lambda_1G_{li})+\frac{1}{2n}(\mu_2F_{li}-\mu_1G_{li}).
\end{aligned}
\tag{3.7}
$$

By forming $F_{jk}(3.5)+G_{jk}(3.6)+H_{jk}(3.7)$, we obtain

$$
Q_{jkl}{}^t\rho_{it}-Q_{jki}{}^t\rho_{lt}=9k(Q_{jkl}{}^tu_{it}-Q_{jki}{}^tu_{lt})+\frac{9}{4n}K_{li}{}^{rh}Q_{jkh}{}^tu_{rt}+A_{jkli},
\tag{3.8}
$$

where we have put

$$
\begin{aligned}
A_{jkli}&=(\frac{9k}{n}\lambda_1+\frac{1}{2n}\mu_1)(G_{jk}H_{li}-H_{jk}G_{li})\\
&\quad+(\frac{9k}{n}\lambda_2+\frac{1}{2n}\mu_2)(H_{jk}F_{li}-F_{jk}H_{li})\\
&\quad+(\frac{9k}{n}\lambda_3+\frac{1}{2n}\mu_3)(F_{jk}G_{li}-G_{jk}F_{li}).
\end{aligned}
$$

Substituting (3.8) into (3.4) and taking account of

$$
A_{lkij}+A_{jlik}+A_{ikjl}+A_{ijlk}+2A_{iljk}+2A_{jkil}=0,
$$

we obtain

$$
K_{jkl}{}^tu_{it}+K_{ilk}{}^tu_{jt}+K_{lij}{}^tu_{kt}+K_{kji}{}^tu_{lt}
$$

$$(3.9)\quad \begin{aligned} &= k[(Q_{lki}{}^t u_{jt} - Q_{lkj}{}^t u_{it}) + (Q_{jli}{}^t u_{kt} - Q_{jlk}{}^t u_{it}) \\ &\quad + (Q_{ikj}{}^t u_{lt} - Q_{ikl}{}^t u_{jt}) + (Q_{ijl}{}^t u_{kt} - Q_{ijk}{}^t u_{lt}) \\ &\quad + 2(Q_{ilj}{}^t u_{kt} - Q_{ilk}{}^t u_{jt}) + 2(Q_{jki}{}^t u_{lt} - Q_{jkl}{}^t u_{it})] \\ &\quad + \frac{1}{4n} T_{lk}{}^r{}_{ij}{}^t u_{rt}, \end{aligned}$$

where we have put

$$(3.10)\quad \begin{aligned} T_{lk}{}^r{}_{ij}{}^t &= Q_{lk}{}^{rh} K_{ijh}{}^t + Q_{jl}{}^{rh} K_{ikh}{}^t + Q_{ik}{}^{rh} K_{jlh}{}^t + Q_{ij}{}^{rh} K_{lkh}{}^t \\ &\quad + 2Q_{il}{}^{rh} K_{jkh}{}^t + 2Q_{jk}{}^{rh} K_{ilh}{}^t. \end{aligned}$$

We shall write (3.9) as

$$(3.11)\quad \begin{aligned} &(K_{jkl}{}^t \delta_i{}^r + K_{ilk}{}^t \delta_j{}^r + K_{lij}{}^t \delta_k{}^r + K_{kji}{}^t \delta_l{}^r) u_{rt} \\ &= k B_{lki}{}^t{}_j{}^r u_{rt} + \frac{1}{4n} T_{lk}{}^r{}_{ij}{}^t u_{rt}, \end{aligned}$$

where we have put

$$\begin{aligned} B_{lki}{}^t{}_j{}^r &= (Q_{lki}{}^t \delta_j{}^r - Q_{lkj}{}^t \delta_i{}^r) + (Q_{jli}{}^t \delta_k{}^r - Q_{jlk}{}^t \delta_i{}^r) \\ &\quad + (Q_{ikj}{}^t \delta_l{}^r - Q_{ikl}{}^t \delta_j{}^r) + (Q_{ijl}{}^t \delta_k{}^r - Q_{ijk}{}^t \delta_l{}^r) \\ &\quad + 2(Q_{ilj}{}^t \delta_k{}^r - Q_{ilk}{}^t \delta_j{}^r) + 2(Q_{jki}{}^t \delta_l{}^r - Q_{jkl}{}^t \delta_i{}^r). \end{aligned}$$

For every point p of the manifold, if $u_{ij}(p)$ takes any skew symmetric value c_{ij}, then (3.11) reduces to

$$(3.12)\quad \begin{aligned} &K_{jkl}{}^t \delta_i{}^r + K_{ilk}{}^t \delta_j{}^r + K_{lij}{}^t \delta_k{}^r + K_{kji}{}^t \delta_l{}^r \\ &- K_{jkl}{}^r \delta_i{}^t - K_{ilk}{}^r \delta_j{}^t - K_{lij}{}^r \delta_k{}^t - K_{kji}{}^r \delta_l{}^t \\ &= k(B_{lki}{}^t{}_j{}^r - B_{lki}{}^r{}_j{}^t) + \frac{1}{4n}(T_{lk}{}^r{}_{ij}{}^t - T_{lk}{}^t{}_{ij}{}^r). \end{aligned}$$

Taking the skew symmetric part of (3.10) with respect to r and t and contracting r and i, we obtain

$$(3.13)\quad T_{lk}{}^r{}_{rj}{}^t - T_{lk}{}^t{}_{rj}{}^r = 0$$

because of

$$Q_{hkr}{}^t K_{jl}{}^{hr} = 3K_{jlk}{}^t + 8kQ_{jlk}{}^t.$$

Contracting with respect to r and i in (3.12) and taking account of (3.13), we obtain

$$\begin{aligned} &(4n-1) K_{jkl}{}^t - 4k(n+2) g_{kl} \delta_j{}^t + 4k(n+2) g_{jl} \delta_k{}^t \\ &= k[(1-4n)(Q_{jlk}{}^t - Q_{klj}{}^t - 2Q_{kjl}{}^t) + 9(-g_{kl}\delta_j{}^t + g_{jl}\delta_k{}^t)], \end{aligned}$$

that is,

$$K_{kjih}=k(g_{kh}g_{ji}-g_{ki}g_{jh}+Q_{khji}-Q_{kijh}-2Q_{kjih}).$$

Thus, by theorem A stated in the section 1, theorem 2 is proved.

§4. A sufficient condition for a tensor to be a Q-Killing tensor.

Let u_{kl} be a Q-Killing tensor. Then we can get

$$\nabla^j\nabla_j u_{kl}=4k(n+2)u_{lk}-K_{krlt}u^{rt}-\rho_{kl}+\frac{1}{3}(Q_{rtkl}+Q_{ktrl})\rho^{rt}, \tag{4.1}$$

by transvection (2.6) with g^{ij}.

In this section we shall show that a skew symmetric tensor u_{kl} satisfying (4.1) is a Q-Killing tensor if there exists a conformal Q-Killing tensor associated with u_{kl} provided that the manifold is compact.

Define a tensor B_{jkl} by

$$B_{jkl}=\nabla_j u_{kl}+\nabla_k u_{jl}-\frac{1}{3}(Q_{jtkl}+Q_{ktjl})\rho^t \tag{4.2}$$

for a skew symmetric tensor u_{kl}, where ρ_j is given by $\nabla^r u_{rj}=\rho_j$.

Simple computation gives us the following equations:

$$\nabla^j B_{jkl}=\nabla^j\nabla_j u_{kl}-4k(n+2)u_{lk}+K_{krlt}u^{rt}+\rho_{kl}-\frac{1}{3}(Q_{rtkl}+Q_{ktrl})\rho^{rt}, \tag{4.3}$$

$$B_{jkl}B^{jkl}=2B_{jkl}\nabla^j u^{kl}-\frac{1}{3}B^{rkl}(Q_{rtkl}+Q_{ktrl})\rho^t, \tag{4.4}$$

where we have used the relation $\nabla_j\nabla_k u^{jk}=0$ and the relation

$$\nabla^j\nabla_k u_{jl}=\rho_{kl}-4k(n+2)u_{lk}+K_{krlt}u^{rt}.$$

Suppose that there exists a conformal Q-Killing tensor s_{kl} associated with u_{kl} by the condition [3]

$$\nabla_j s_{kl}+\nabla_k s_{jl}=2\xi_l g_{jk}-\xi_k g_{jl}-\xi_j g_{kl}+3(Q_{jtkl}+Q_{ktjl})\nabla^r u_r{}^t. \tag{4.5}$$

If we put $v_{kl}=-\frac{1}{9}s_{kl}$, then by virtue of (4.2) and (4.5), it follows that

$$2B_{jkl}\nabla^j v^{kl}=-\frac{1}{3}B^{rkl}(Q_{rtkl}+Q_{ktrl})\rho^t. \tag{4.6}$$

Taking account of (4.4) and (4.6), we obtain

$$B_{jkl}B^{jkl}=2B_{jkl}\nabla^j(u^{kl}+v^{kl}). \tag{4.7}$$

Substituting (4.3) and (4.7) into

$$\nabla^j[B_{jkl}(u^{kl}+v^{kl})]=(\nabla^j B_{jkl})(u^{kl}+v^{kl})+B_{jkl}\nabla^j(u^{kl}+v^{kl}),$$

and making use of theorem of Green, we obtain, provided that the manifold is compact,

$$\int_M[(u^{kl}+v^{kl})\{\nabla^j\nabla_j u_{kl}-4k(n+2)u_{lk}+K_{krlt}u^{rt}+\rho_{kl}$$

$$-\frac{1}{3}(Q_{rtkl}+Q_{ktrl})\rho^{rt}\}+\frac{1}{2}B_{jkl}B^{jkl}]d\sigma=0,$$

where $d\sigma$ denotes the volume element of the manifold and $\rho_{kl}=\nabla_k\nabla^r u_{rl}$.

Thus we have the following

THEOREM 3. *In a compact orientable quaternion Kaehlerian manifold M, if the relation* (4.1) *is satisfied for a skew symmetric tensor field* u_{kl} *and there exists in M a conformal Q-Killing tensor associated with* u_{kl}, *then* u_{kl} *is a Q-Killing tensor.*

In a previous paper [2], we showed that the covariant derivative of a Killing vector ξ_l in a quaternion Kaehlerian manifold of constant Q-sectional curvature $4k$ satisfies the following condition:

$$\nabla_j\nabla_k\xi_l+\nabla_k\nabla_j\xi_l=-k[2\xi_l g_{jk}-\xi_k g_{jl}-\xi_j g_{kl}+3(Q_{jtkl}+Q_{ktjl})\xi^t]. \tag{4.8}$$

Using $-\frac{1}{k}\nabla_k\xi_l$ instead of s_{kl} in (4.5), considering ξ_l instead of $\nabla^r u_{rl}$ and taking account of theorem 1 and theorem 3, we have the following

THEOREM 4. *In a compact orientable quaternion Kehlerian manifold of constant Q-sectional curvature, a necessary and sufficient condition for a skew symmetric tensor field* u_{kl} *to be a Q-Killing tensor is that it satisfies the relation* (4.1) *and the associated vector of* u_{kl} *is a Killing vector.*

§ 5. *Q*-Killing tensor in a quaternion Kaehlerian manifold of constant *Q*-sectional curvature.

In this section we consider a Q-Killing tensor in a quaternion Kaehlerian manifold of constant Q-sectional curvature.

For an arbitrary Q-Killing tensor u_{kl}, the covariant derivative of the associated vector $\xi_l=\nabla^r u_{rl}$ of u_{kl} satisfies condition (4.8) by virtue of theorem 1. If we put

$$9u_{kl}+\frac{1}{k}\nabla_k\xi_l=v_{kl},$$

then v_{kl} is a conformal Killing tensor whose associated vector is $-\xi_l$, that is,

$$\nabla_j v_{kl}+\nabla_k v_{jl}=-(2\xi_l g_{jk}-\xi_k g_{jl}-\xi_j g_{kl}).$$

On the other hand, it is well known that the covariant derivative of a Killing vector ξ_l is closed. Consequently an arbitrary Q-Killing tensor u_{kl} is decomposed as

(5.1) $$u_{kl}=p_{kl}+q_{kl},$$

where p_{kl} is a conformal Killing tensor and $q_{kl}=-\frac{1}{9k}\nabla_k\xi_l$ is a closed conformal Q-Killing tensor. (In [3], we defined a conformal Q-Killing tensor as a skew symmetric tensor field s_{kl} satisfying (4.5).) Moreover the uniqueness of this decomposition follows from the following

LEMMA. *In a quaternion Kaehlerian manifold of constant Q-sectional curvature, if a conformal Killing tensor is closed, then it is a zero tensor.*

Proof. Let ξ_{jk} be a closed conformal Killing tensor, then we have

$$\nabla_k\xi_{jl}+\nabla_l\xi_{kj}+\nabla_j\xi_{lk}=0,$$

$$\nabla_k\xi_{jl}+\nabla_j\xi_{kl}=2\rho_l g_{kj}-\rho_j g_{kl}-\rho_k g_{jl}.$$

Combining these two equations, we easily see that

(5.2) $$\nabla_l\xi_{jk}=\rho_k g_{jl}-\rho_j g_{kl}.$$

Operating ∇_h to (5.2) and making use of Ricci identity, we find

(5.3) $$K_{hlj}{}^t\xi_{tk}+K_{hlk}{}^t\xi_{jt}=\rho_{hj}g_{kl}-\rho_{hk}g_{jl}+\rho_{lk}g_{jh}-\rho_{lj}g_{kh},$$

where we have put $\rho_{hj}=\nabla_h\rho_j$.

Since a quaternion Kaehlerian manifold is an Einstein space, the associated vector ρ_k of a conformal Killing tensor is a Killing vector (theorem 4 of [6]), that is,

(5.4) $$\rho_{hj}+\rho_{jh}=0.$$

Transvecting (5.3) with g^{lk} and taking account of (5.4), we obtain

(5.5) $$(4n-2)\rho_{hj}=K_{hrjt}\xi^{tr}+4k(n+2)\xi_{jh}.$$

Substituting (5.5) into the right hand member of (5.3), we get

$$(1-\frac{1}{4n-2})(K_{hlj}{}^t\xi_{kt}+K_{lhk}{}^t\xi_{jt})$$

(5.6) $$=\frac{1}{4n-2}[4k(n+2)(\xi_{hk}g_{jl}-\xi_{hj}g_{kl}-\xi_{lk}g_{jh}+\xi_{jl}g_{kh})$$

$$+(K_{lrjt}g_{kh}-K_{lrkt}g_{jh})\xi^{tr}].$$

Since a quaternion Kaehlerian manifold is an Einstein space, by virtue of (2.11) of [6], we can obtain following relation for a conformal Killing tensor ξ_{lk}:

(5.7) $$K_{jki}{}^t\xi_{lt}+K_{ilj}{}^t\xi_{kt}+K_{lik}{}^t\xi_{jt}+K_{kjl}{}^t\xi_{it}=0.$$

Substituting (5.6) into (5.7), we get

$$8k(n+2)(\xi_{hk}g_{jl}+\xi_{jl}g_{kh})+(K_{lrjt}g_{kh}+K_{krht}g_{lj})\xi^{tr}=0.$$

Transvecting this equation with g^{hk}, we obtain

$$8k(n+2)\xi_{jl}+K_{lrjt}\xi^{tr}=0. \tag{5.8}$$

Substituting (1.8) into (5.8), we find

$$Q_{rtlj}\xi^{rt}=(8n+17)\xi_{jl}-Q_{lrjt}\xi^{rt}. \tag{5.9}$$

Transvecting (5.9) with $Q_k{}^l{}_i{}^j$, and taking account of

$$Q_k{}^l{}_i{}^jQ_{rtlj}=Q_{kitr},\quad Q_k{}^l{}_i{}^jQ_{lrjt}=3g_{kr}g_{it}+2Q_{krit},$$

we get

$$(8n+19)Q_{ltjr}\xi^{rt}=3\xi_{lj}-Q_{rtlj}\xi^{rt}. \tag{5.10}$$

Substituting (5.10) into the right hand member of (5.9), we obtain

$$Q_{rtlj}\xi^{rt}=8(n+2)\xi_{jl},$$

and this equation is written as

$$8(n+2)\xi_{jl}+(\nu_1F_{jl}+\nu_2G_{jl}+\nu_3H_{jl})=0, \tag{5.11}$$

where we have put

$$\nu_1=F_r{}^t\xi_t{}^r,\quad \nu_2=G_r{}^t\xi_t{}^r,\quad \nu_3=H_r{}^t\xi_t{}^r.$$

Transvecting (5.11) with F^{lj}, we easily see that $\nu_1=0$. Similarly we obtain $\nu_2=0$ and $\nu_3=0$. Therefore we have from (5.11)

$$\xi_{jl}=0.$$

Thus the lemma is proved.

Hence we have the following

THEOREM 5. *In a quaternion Kaehlerian manifold of constant Q-sectional curvature with* $4k$ $(k=K/16n(n+2)\neq 0)$, *a Q-Killing tensor* u_{kl} *is uniquely decomposed in the form*

$$u_{kl}=p_{kl}+q_{kl},$$

where p_{kl} *is a conformal Killing tensor and* q_{kl} *a closed conformal Q-Killing tensor. In this case* q_{kl} *is of the form*

$$q_{kl}=(-1/9k)\nabla_k\xi_l,$$

where ξ_l *is the associated vector of* u_{kl}.

Conversely if p_{kl} *is a conformal Killing tensor whose associated vector is* $\frac{1}{9}(4n-1)\xi_l$ *and* ξ_l *is a Killing vector, then* u_{kl} *given by* (5.1) *is a Q-Killing tensor.*

References

[1] C-H. Chen, *On F-Killing tensors in a Kaehlerian space*, Bull. of College of Naval Architecture of Nagasaki, **12** (1971), 53-56.

[2] S-S. Eum, *On Q-conformal Killing tensors in a quaternion Kaehlerian manifold*, Sung Kyun Kwan University Journal **21** (1975), 43-48.

[3] ________, *On conformal Q-Killing tensors in a quaternion Kaehlerian manifold*, Journal of Korean Math. Soc., **13** (1976), 39-49.

[4] S. Ishihara, *Quaternion Kaehlerian manifolds*, Journal of Differential Geometry, **9** (1974), 483-500.

[5] S. Tachibana, *On Killing tensors in a Riemannian space*, Tōhoku Math. Journal, **20** (1968), 257-264.

[6] ________, *On conformal Killing tensors in a Riemannian space*, Tōhoku Math. Journal, **21** (1969), 56-64.

[7] K. Yano, S-S. Eum and U-H. Ki, *On almost contact affine 3-structures*, Kōdai Math. Sem. Rep., **25** (1973), 129-142.

Sung Kyun Kwan University