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ON THE CLASS OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES HAVING FINITE ORDER POINT

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§ 1 Preliminaries.

The topological aspect of the expansion theory of various spaces of entire functions have been developed in ([1], [4], [5], [6], [7], [8]). Towards our aim, we single out our recent work in [7], where we have obtained a few topological structuaral properties of the space Y of entire functions of two complex variables having finite order point. Our aim in this note is to explore further such properties for the space Y. Indeed we prove that Y is a perfect or a semi-Montel space via characterizing bounded sets in Y and thereby adding an example of a Montel space to the literature of locally convex spaces.

We now come to defining various terms and notations which we will be using here. Let $\boldsymbol{C}$ be the finite complex plane equipped with its usual topology and Y be the space of entire functions f: $\boldsymbol{C}^2 \to \boldsymbol{C}$, having an order point at most equal to (ρ_1, ρ_2), where ρ_1 and ρ_2 are preassigned positive finite numbers. In other words, Y consists of all functions f which satisfy

(1.1) $$f(z_1, z_2) = \sum_{m+n \geqq 0} \sum a_{mn} z_1^m z^n_2, \text{ for all } (z_1, z_2) \in \boldsymbol{C}^2,$$

with

(1.2) $$\limsup_{m+n \to \infty} |a_{mn}|^{\frac{1}{m+n}} = 0; \text{ and}$$

(1.3) $$M(f; r_1, r_2) \leqq \exp[r_1^{\rho_1+\varepsilon} + r_2^{\rho_2+\varepsilon}],$$

valid for all large r_1, r_2 and for every $\varepsilon > 0$, where

$$M(f;\ r_1, r_2) = \sup_{|z_1| \leqq r_1, |z_2| \leqq r_2} |f(z_1, z_2)|, \quad r_1, r_2 > 0.$$

According to [7], a characterization of members of Y can also be given in terms of the coefficients occuring in the expansion, as follows:

THEORM 1.1. *A function f having representation* (1.1) *is in Y if and only if for each $\delta > 0$, there exists an integer $N_0 = N_0(\delta)$ such that*

(1.4) $$|a_{mn}| \leqq m^{-\frac{m}{(\rho_1+\delta)}} {}^{-\frac{n}{(\rho_2+\delta)}},$$

for all $m+n \geqq N_0$.

The above characterization helps us in defining a a locally convex topology τ on Y in

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a natural way by means of the family $\{\|f;\ \rho_1+\delta,\ \rho_2+\delta\|\ : \delta>0\}$ of norms on Y, where

$$(1.5)\qquad \|f;\ \rho_1+\delta, \rho_2+\delta\| = |a_{00}| + \sum_{m+n\geq 1}\sum |a_{mn}|\, m^{\frac{m}{(\rho_1+\delta)}} n^{\frac{n}{(\rho_2+\delta)}}$$

The topology τ can also be shown to have been generated by an invariant metric d given by the Frechet combination of norms, as follows:

$$(1.6)\qquad d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f-g;\rho_1+\frac{1}{k},\ \rho_2+\frac{1}{k}\|}{1+\|f-g;\rho_1+\frac{1}{k},\ \rho_2+\frac{1}{k}\|}.$$

We assume, throughout, that the space Y is equipped with the topology τ and the terminology is from [2]. We now mention a few results, proved in [7], which will be needed in the sequel.

THEOREM 1.2: *Y is a Frechet space (i.e., a locally convex complete metrizable topological vector space).*

THEOREM 1.3: *Every continuous linear functional ϕ on Y, i.e., $\phi \in Y^*$ (topological dual of Y), is of the form*

$$\phi(f) = \sum_{m+n\geq 0}\sum c_{mn} a_{mn},$$

where f is given by (1.1), if and only if

$$|c_{mn}| \leq K\ m^{m/(\rho_1+\delta)} n^{n/(\rho_2+\delta)},$$

holds for all $m, n\geq 0$, for some constant $K>0$ and some $\delta>0$.

§ 2 Characterization of Bounded Sets in Y.

As the bounded sets in a locally convex space play a vital role in deciding whether the space is a Montel space or not, we characterize them here for the space Y, in terms of the coefficients occuring in the expansion, We have already proved in [7] that the sequence $\{\delta_{mn}: m, n\geq 0\}$, where $\delta_{mn}(z_1, z_2) = z_1^m z_2^n$, forms a base for Y with respect to its Frechet topology and consequently it follows from a result of Newns [10] that the coefficients a_{mn}, $m, n\geq 0$, which occur in the representation of each $f\in Y$, are continuous linear functionals on Y. Thus $a_{mn}\equiv a_{mn}(f)\in Y^*$, for all $m, n\geq 0$. Hence, for each $f\in Y$, (1, 1) can be written as:

$$(2.1)\qquad f = \sum_{m+n\geq 0}\sum a_{mn}(f)\delta_{mn},$$

the convergence of the double infinite series being taken with respect to the topology τ on Y. We now prove:

THEOREM 2.1. *A subset B of Y is bounded in Y if and only if for every $\delta>0$, there exists a constant $M>0$ such that*

$$|a_{mn}(f)| \leqq M m^{-\frac{m}{(\rho_1+\delta)}} n^{-\frac{n}{(\rho_2+\delta)}}, \tag{2.2}$$

for all $m, n \geqq 0$ and evevry $f \in A$.

Proof. Let B be a bounded subset of Y. Then it is obviously bounded with respect to each norm $\|f;\ \rho_1+\delta,\ \rho_2+\delta\|$, $\delta > 0$. Therefore, for every $\delta > 0$, there exists a constant $M > 0$ such that

$$\|f;\ \rho_1+\delta,\ \rho_2+\delta\| \leqq M, \text{ for every } f \in B,$$

i. e.

$$|a_{00}| + \sum_{m+n \geqq 1}\sum |a_{mn}(f)| m^{\frac{m}{(\rho_1+\delta)}} n^{\frac{n}{(\rho_2+\delta)}} \leqq M.$$

This clearly implies (2.2).

To prove the converse, let us assume that the condition (2.2) is true. For $\phi \in Y^*$, we have from Theorem 1.3, that

$$\phi(f) = \sum_{m+n \geqq 0}\sum a_{mn}(f) c_{mn},$$

where

$$|c_{mn}| \leqq K\ m^{\frac{m}{(\rho_1+\delta)}} n^{\frac{n}{(\rho_2+\delta)}},$$

for every $m, n \geqq 0$, some $K > 0$ and some $\delta > 0$. Choose $0 < \eta < \delta$. Then, by hypothesis, there exists a constant $M = M(\eta)$ such that

$$|a_{mn}(f)| \leqq m^{-\frac{m}{(\rho_1+\delta)}} n^{-\frac{n}{(\rho_2+\eta)}},$$

for every $m, n \geqq 0$ and every $f \in B$. Now, for any $f \in B$, consider

$$|\phi(f)| = |\sum_{m+n \geqq 0}\sum a_{mn} c_{mn}|$$

$$\leqq \sum_{m+n \geqq 0}\sum |a_{mn}|\,|c_{mn}|$$

$$\leqq MK \sum_{m+n \geqq 0}\sum m^{\frac{m(\eta-\delta)}{(\rho_1+\delta)(\rho_2\eta)}} n^{\frac{n(\eta-\delta)}{(\rho_2+\delta)(\rho_2+\eta)}}.$$

As the series on the right hand side is convergent, it follows that

$$|\phi(f)| \leqq C, \text{ for every } f \in B,$$

where C is a constant depending on ϕ. Hence B is $\sigma(Y, Y^*)$-bounded and therefore, bounded in the topology τ, by Mackcy's Theorem [2, p. 209]. This completes the proof.

THEOREM 2.2. *The sequence $\{a_{mn}\} \subset Y^*$ of coefficient functionals associated with the Schauder base $\{\delta_{mn}\} \subset Y$ is $\beta(Y^*, Y)$-bounded (i.e., strongly bounded in Y^*).*

Proof. As the topology $\beta(Y^*, Y)$ is generated by the polars of all $\sigma(Y, Y^*)$-bounded sets in Y, consider any arbitrary but fixed $\sigma(Y, Y^*)$-bounded set B in Y. Then by Mackey's Theorem, B is τ-bounded and therefore, for any given $\eta>0$, we have from the previous theorem that

$$|a_{mn}(f)| \leqq M m^{-\frac{m}{(\rho_1+\delta)}} n^{-\frac{n}{(\rho_1+\delta)}},$$

for all $m, n \geqq 0$and every $f \in B$. Hence,

$|a_{mn}(f)| \leqq M$, for every $m, n \geqq 0$ and $f \in B$, because for $m, n \geqq 1$, we have $m^{-m/(\rho_1+\delta)} \leqq 1$, $n^{-n/(\rho_2+\delta)} \leqq 1$, and for $m, n=0$, we assume the value of such an expression equal to 1. This shows that $\{a_{mn}: m, n \geqq 0\} \subset MB^0$, thereby proving that $\{a_{mn}: m, n \geqq 0\}$ is bounded in the strong topology.

Alternative proof

$$|a_{mn}(f)| < m^{-\frac{m}{(\rho_1+\delta)}} {}^{-\frac{n}{(\rho_2+\delta)}},$$

for $m+n \geqq N_0$, where N_0 is an integer depending on δ, for each $\delta>0$. We fix $\delta>0$. Then

$$|a_{mn}(f)| \leqq 1,$$

for $m+n \geqq N_0$. For $m+n \leqq N_0$, choose M_f such that

$$M_f = \sup_{0 \leqq m+n \leqq N_0} |a_{mn}(f)|,$$

and write $K_f = \max(1, M_f)$. Then

$$|a_{mn}(f)| \leqq K_f,$$

for every $f \in Y$ and all $m, n \geqq 0$. Hence the set $\{a_{mn}: m, n \geqq 0\}$ is $\sigma(Y^*, Y)$-bounded (i. e. weakly bounded). The space Y, being a Fréchet space, is barrelled ([2], p. 214). Therefore, it follows that the set $\{a_{mn}: m, n \geqq 0\}$ is strongly bounded (see Corollary, p. 212, [2]). This completes the proof.

Since in the dual of a barrelled space, the weakly-bounded, strongly bounded and equicontinuous subsets are same ([2], p. 212), we also have:

COROLLARY 2. 3. *The sequence $\{a_{mn}: m, n \geqq 0\}$ associated with the Schauder base $\{\delta_{mn}\} \subset Y$ is equicontinuous on Y.*

From a result of Kalton (Prop. 2. 1, p. 92, [3]) and the above corollary, we derive a result proved in [1] for the base $\{\delta_{mn}\}$ for the space of analytic functions in bicylinders.

COROLLARY 2. 4. *The base $\{\delta_{mn}\}$ is a regular base (that is, there exists a τ-neighbourhood G of $0 \in Y$ such that $\delta_{mn} \notin G$, for all $m, n \geqq 0$).*

§ 3. The Main Result.

Let us recall that a locally convex topological vector space $(lC\ TVS)\chi$ is a Montel

-space if it is an infrabarrelled space ([2], p. 231) and that every bounded set in χ is relatively compact. For proving our main result, we need a few morer esults which are:

LEMMA 3.1. *Let χ be a vector space equipped with a countable number of semi-norms $\{p_k\}$. Define*

$$\lambda(f,g)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{p_k(f-g)}{1+p_k(f-g)},\ \ for\ f,g\in\chi.$$

Suppose for $\varepsilon>0$ and $f,g\in\chi$, $f\neq 0$, $\lambda(f,g)\varepsilon$. X Then there exists an integer k_0, such that $P_{k_0}(f-g)>\frac{\varepsilon}{2-\varepsilon}$.

Proof. See [8], for thr proof.

LEMMA 3.2. *Let B be a bounded set in Y and ε and δ be any two arbitrary positive numbers. Then there exists a positive integer $N=N(\varepsilon,\delta)$ such that*

$$\|\sum_{m+n>N}\sum a_{mn}(f)\delta_{mn};\ \rho_1+\delta,\ \rho_2+\delta\|<\varepsilon,$$

uniformly in $f\in B$.

Proof. Choose, η, $0<\eta<\delta$. Then by Theorem 2.1, there exists a constant $M>0$, where M depends on η and therefore on δ, such that

$$|a_{mn}|\leqq m^{-\frac{m}{(\rho_1+\eta)}}n^{-\frac{n}{(\rho_2+\eta)}},$$

for all $m,n\geqq 0$ and every $f\in B$. Therefore we have

$$\|\sum_{m+n\geqq 0}\sum a_{mn}(f)\delta_{mn};\ \rho_1+\delta,\ \rho_2+\delta\|$$

$$=\sum_{m+n\geqq 0}\sum|a_{mn}(f)|m^{\frac{m}{(\rho_1+\delta)}}n^{\frac{n}{(\rho_2+\delta)}}$$

$$\leqq M\sum_{m+n\geqq 0}\sum m^{\frac{(\eta-\delta)m}{(\rho_1+\delta)(\rho_1+\delta)}}n^{\frac{n(n-\delta)}{(\rho_2+\eta)(\rho_2+\delta)}}$$

for every $f\in B$. Since $0<\eta<\delta$, the series on the right hand side is convergent and can be made as small as ε is by choosing $m+n$ sufficiently large. As M depends on δ, the required result follows easily.

Combining lemmas 3.1 and 3.2, we get

LEMMA 3.3. *Let B be a bounded set in Y. Then for $\varepsilon>0$, there exists an integer $N(\varepsilon)$ such that*

$$d(\sum_{m+n\geqq N(\varepsilon)}\sum a_{mn}(f)\ \delta_{mn},0)<\varepsilon,$$

for every $f\in B$.

We now state and prove our main result, namely,

THEOREM 3.4. *The space Y is a Montel space.*

Proof. As Y is a Fréchet space and therefore, infrabarrelled, it is sufficient to prove that every bounded set is relatively compact. Let B be a bounded subset of Y. Then for $\varepsilon>0$, there exists an integer $N_0=N_0(\varepsilon)$, such that

$$d(\sum_{m+n\geq N_0(\varepsilon)}\sum a_{mn}(f)\delta_{mn}, 0)<\frac{\varepsilon}{2}$$

for every $f\in B$, where $f=\sum_{m+n\geq 0}\sum a_{mn}(f)\delta_{mn}$.

Consider

$$S_{N_0}=\{\sum_{0<m+n\leq N_0}\sum a_{mn}(f)\delta_{mn};\ f\in B\}$$

The set S_{N_0} is contained in the finite dimensional space generated by $\{\delta_{mn}:\ 0\leq m+n\leq N_0\}$ and is bounded by Theorem 2.1. Therefore, it is relatively compact, Hence, there exists a finite $\frac{\varepsilon}{2}$-net E for S_{N_0}. This set E serves as a finite ε-net for the set B, because if $f\in B$ with $f=\sum_{m+n\geq 0}\sum a_{mn}(f)\delta_{mn}$, then $g=\sum_{0\leq m+n\leq N_0}\sum a_{mn}(f)\delta_{mn}$ is in S_{N_0}. For $g\in S_{N_0}$, there exists a $g^*\in E$ such that

$$d(g, g^*)<\frac{\varepsilon}{2}$$

$$d(f, g^*)\leq d(f, g)+d(g, g^*)=d(f-g, 0)+d(g, g^*)$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus for every $\varepsilon<0$, these exists a finite ε-net for the set B. Hence the set B is relatively compact (see [9], p. 57). The proof is now complete.

As every Montel space Y is a reflexive ([2], p. 231), we derive

COROLLARY 3.5: *The space Y is a reflexive space*

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