J. Korean Math. Soc.
Vol. 13, No. 1, 1976.

FOX H-FUNCTION AND THE TEMPERATURE IN A SLAB WITH FACES AT TEMPERATURE ZERO

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1. Introduction.

In recent years a number of authors have used Meijer G-function, H-function of Fox in heat conduction problems of bar, cylinder etc [1, 5].

The H-function introduced by Fox [4, p. 408] will be represented and defined as follows:

$$(1.1)\qquad H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}\{(a_p, e_p)\}\\ \{(b_q, f_q)\}\end{matrix}\right]=\frac{1}{2\pi i}\int_L\frac{\prod^m_{j=1}\Gamma(b_j-f_js)\prod^n_{j=1}\Gamma(1-a_j+e_js)z^s}{\prod^q_{j=m+1}\Gamma(1-b_j+f_js)\prod^p_{j=m+1}\Gamma(a_j-e_js)}ds$$

Where z is not equal to zero and an empty product is interpreted as unity; p, q, m and n are integers satisfying $0\leqq m\leqq q$, $0\leqq n\leqq p$; $e_j(j=1, 2, \cdots p)$, $f_h(h=1, \cdots q)$ are positive numbers and $a_j(j=1, \cdots p)$, $b_h(h=1, \cdots q)$ are complex numbers. L is the path of intgration separating the increasing and decreasing sequences of the poles of the integrand. The parameters $\{(a_p, e_p)\}$ represent $(a_1, e_1)\cdots(a_p, e_p)$. These assumptions for the H-function will be adhered to through-out this paper.

As an example of the application of H-function of Fox in applied mathematic, we shall consider the problem of determining the temperature in a slab of homogeneous material bounded by the planes $x=0$ and $x=\pi$ having an initial temperature $u=f(x)$, varing only with the distances from the faces and with its two faces kept at zero temperature.

The formula for the temperature u at any instant and at all points of the slab is to be determined. In the problem it is clear that the temperature function is of the variables x and t only. Hence at each interior point this function $u(x, t)$ must satisfy the heat equation for one dimensional form.

$$(1.2)\qquad \frac{\partial u}{\partial t}=K\frac{\partial^2 u}{\partial x^2}\ (0<x<\pi,\ \ t>0)$$

In addition, it must satisfy the conditions.

$$(1.3)\qquad u(+0, t)=0$$

$$u(\pi-0, t)=0 \qquad (t>0)$$

$$(1.4)\qquad u(x, +0)=f(x) \quad (0<x<\pi)$$

The boundary value problem (1.2)—(1.4) is also the problem of temperatures in a right prism or cylinder whose length is π (taken so for conveniences in the computation), provided its later surface is insulated. Its ends $x=0$ and $x=\pi$ are held at tempe-

Received by the editors Nov. 10, 1975.

rature zero and its initial temperature is $f(x)$.

In section 2 of this paper, we have evaluated an integral involving Fox H-function which is required in the proof of subsequent section.

Here a_p denotes $a_1, \cdots\cdots a_p$, ∂ is positive integer and the symbol $\Delta(\partial, a)$ represents the set of parameter $\frac{a}{\partial}, \frac{a+1}{\partial}, \cdots\cdots\cdots \frac{a+\partial-1}{\partial}$.

In this paper we shall consider.

$$(1.5) \qquad f(x)=(\sin\tfrac{x}{2})^{2n-\beta-1}(\cos\tfrac{x}{2})^{\beta-1}H_{p,q}^{m,n}\left[z(\tan\tfrac{x}{2})^{2\partial}\,\Big|\,\begin{matrix}\{(a_p,e_p)\}\\ \{(b_q,f_q)\}\end{matrix}\right]$$

2. The integral.

$$(2.1) \qquad \int_0^\pi \sin n\theta(\sin\tfrac{\theta}{2})^{2n-\beta-1}(\cos\tfrac{\theta}{2})^{\beta-1}H_{p,q}^{m,n}\left[z(\tan\tfrac{\theta}{2})^{2\partial}\,\Big|\,\begin{matrix}\{(a_p,e_p)\}\\ \{(b_q,f_q)\}\end{matrix}\right]d\theta$$

$$=\frac{(2\partial)^{2n-1}}{(2\pi)^{\partial-1}\Gamma(2n)}H_{p+2\partial,q+2\partial}^{m+2\partial,n+\partial}\left[z\,\Big|\,\begin{matrix}\Delta(\partial,\frac{1+\beta-2n}{2}), & (a_p,e_p), & \Delta(\partial,\frac{2+\beta-2n}{2})\\ \Delta(2\partial,\beta), & (b_q,f_q) & \end{matrix}\right]$$

where

$$\phi=\sum_{j=1}^{n}e_j-\sum_{j=n+1}^{p}e_j+\sum_{j=1}^{m}f_j-\sum_{j=m+1}^{q}f_j>0,$$

$$|\arg z|<\phi\cdot\frac{\pi}{2}$$

and $2n>\mathrm{Re}(\beta)>0$, $n=1,2,3,\cdots\cdots$

Proof. To establish (2.1) express the H-function as a Mellin Barnes type integral [4, p. 408] and interchange the order of integration, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js)z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=n+1}^{p}\Gamma(a_j-e_js)}ds\times$$

$$\int_0^\pi\sin n\theta\,(\sin\tfrac{\theta}{2})^{2n+2s\partial-\beta-1}(\cos\tfrac{\theta}{2})^{\beta-2s\partial-1}d\theta$$

Now evaluating the inner integral with the help of the modified form of the formula [6] namely.

$$\int_0^\pi\sin n\theta(\sin\tfrac{\theta}{2})^{2n-\beta-1}(\cos\tfrac{\theta}{2})^{\beta-1}d\theta=\frac{2^{2n-\beta}\Gamma(\pi)\Gamma(\frac{2n-\beta+1}{2})\Gamma(\beta)}{\Gamma(1+\frac{\beta}{2}-n)\Gamma(2n),} \qquad (2n>\mathrm{Re}(\beta)>0.)$$

and using the multiplication formula for the gamma function [3, p. 4(11)] we get

$$\frac{(2\partial)^{2n-1}}{(2\pi)^{\partial-1}\Gamma(2n)}\times$$

$$\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js)\prod_{j=0}^{\partial-1}\Gamma\left(\frac{\frac{2n-\beta+1}{2}+i}{\partial}+s\right)\prod_{i=0}^{2\partial-1}\Gamma\left(\frac{\beta+i}{2\partial}-s\right)z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=n+1}^{p}\Gamma(a_j-e_js)\prod_{i=0}^{\partial-1}\Gamma\left(\frac{\frac{2+\beta-2n}{2}+i}{\partial}-s\right)}ds$$

on applying [4, p. 408] the value of the integral (2.1) is obtained.

3. The solution of problem.

The solution of the problem is

$$(3.1)\quad u(x,t)=\frac{4}{(2\pi)^{\partial}}\sum_{s=1}^{\infty}\frac{(2\partial)^{s-1}e^{-s^2kt}\sin sx}{\sqrt{2s}}H^{m+2\partial,\,n+\partial}_{p+2\partial,q+2\partial}\left[z\,\middle|\,\begin{matrix}\Delta(\partial,\frac{1+\beta-2n}{2}),\ (a_p,e_p)\\ \Delta(2\partial,\beta),\quad (b_q,f_q)\end{matrix}\right.$$

$$\left.\Delta(\partial,\frac{2+\beta-2n}{2})\right]$$

where $\phi=\sum_{j=1}^{n}e_j-\sum_{j=n+1}^{p}e_j+\sum_{j=1}^{m}f_j-\sum_{j=m+1}^{q}f_j>0,\quad |\arg z|<\phi\cdot\frac{\pi}{2}$

and $\mathrm{Re}(\beta)>0.$

Proof. The solution of the problem can be written as [2, p. 139(6)]

$$(3.2)\qquad u(x,t)=\sum_{s=1}^{\infty}A_s\exp(-s^2kt)\sin sx$$

If $t=0$, then by virtue of (1.5), we have

$$(3.3)\qquad \left(\sin\frac{x}{2}\right)^{2n-\beta-1}\left(\cos\frac{x}{2}\right)^{\beta-1}H^{m,\,n}_{p,\,q}\left[z\left(\tan\frac{x}{2}\right)^{2\partial}\,\middle|\,\begin{matrix}\{(a_p,e_p)\}\\ \{(b_q,f_q)\}\end{matrix}\right]=\sum_{s=1}^{\infty}A_s\sin sx$$

Multiplying both sides of (3.3) by $\sin nx$ and integrating with respect to x from 0 to π we get

$$(3.4)\qquad \int_0^{\pi}\sin nx\left(\sin\frac{x}{2}\right)^{2n-\beta-1}\left(\cos\frac{x}{2}\right)^{\beta-1}H^{m,\,n}_{p,\,q}\left[z\left(\tan\frac{x}{2}\right)^{2\partial}\,\middle|\,\begin{matrix}\{(a_p,e_p)\}\\ \{(b_q,f_q)\}\end{matrix}\right]dx$$

$$=\sum_{s=1}^{\infty}A_s\int_0^{\pi}\sin sx\sin nx\,dx$$

Now using (2.1) and the orthogonal property of the sine function we have

$$(3.5)\quad A_s=\frac{(2\partial)^{\partial s-1}}{(2\pi)^{\partial}\Gamma(2s)}H^{m+2\partial,\,n+\partial}_{p+2\partial,q+2\partial}\left[z\,\middle|\,\begin{matrix}\Delta(\partial,\frac{1+\beta-2n}{2}),\ (a_p,e_p),\ \Delta(\partial,\frac{2+\beta-2n}{2})\\ \Delta(2\partial,\beta),\quad (b_q,f_q)\end{matrix}\right],$$

with the help of (3. 2) and (3. 5) the solution (3. 1) is obtained.

4. Conclusion.

On specializing the parameter the H-function may be converted into G-function, Bessel function, Legendre function and other higher transcendental functions [3, pp216-222]. Therefore the function $f(x)$ given in (1. 5) is of general character and hence may encompass several cases of interest.

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