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Continuity of linear maps and their transpose

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1. Introduction

This note is devoted to the continuity of linear maps and of their transpose, relative to the several $\mathfrak{S}$—topologies and to the relationship between the continuity of linear maps and their transpose, for which domain and range spaces of the maps equipped with various topologies on the spaces and their dual spaces; the weak, the strong, the Mackey, the Arens topology and the topology of uniform convergence of all precompact subsets of the space. The other results for the relatively strong topology are especially mentioned without the proof of necessary Lemmas. Throughout this note let E and F be two locally convex Hausdorff (topological vector) spaces and E′ and F′ be their topological duals, respectively.

2. $\mathfrak{S}$—topologies and Lemmas.

Let $\mathfrak{S}$ be a family of bounded subsets of E which cover E and let for $A\in\mathfrak{S}$ the polar $A^\circ=\{y\in E' : |\langle x, y\rangle|\leq 1,$ for all $x\in A\}$ of A, then $\mathfrak{S}^\circ=\{A^\circ : A\in\mathfrak{S}\}$ form a collection of absorbing, convex, balanced sets in E′. It is well known that $\mathfrak{B}=\{\lambda(\bigcap_{i=1}^n A_i^\circ) : \lambda>0,\ A_i^\circ\in\mathfrak{S}^\circ\}$ form a basis of neighborhoods (nbh) of zero in a certain locally convex topology on E′. We shall call it the $\mathfrak{S}$—topology or the topology of uniform convergence on sets belonging to $\mathfrak{S}$. Since $\mathfrak{S}$ covers E and E′ is locally convex Hausdorff, the $\mathfrak{S}$—topology on E is Haustorff. Similarly, by symmetric method (E and E′ are replaced by E′ and E, respectively), we can define the $\mathfrak{S}$—topology on E.

DEFINITION 1. Let $\mathfrak{S}$ be the family of all finite subsets of E(E′) then the corresponding $\mathfrak{S}$—topology on E′(E) will be called the weak topology and will be denoted by $w(E', E)$ $(w(E, E'))$. As we know, the collection of $U(x_1'\cdots x_n', \varepsilon)=\{x : |\langle x, x_i'\rangle|\leq\varepsilon,\ x_i'\in E$ for $i=1, \cdots, n\}$ form the basis of nbhs of zero for the weak topology $w(E, E')$ on E. Similarly $U(x_1\cdots x_n, \varepsilon)=|x' : |\langle x_i, x'\rangle|\leq\varepsilon,\ x_i\in E$ for $i=1, 2, \cdots, n\}$ is a nbh of zero for the weak topology $w(E', E)$ on E′.

DEFINITION 2. If $\mathfrak{S}$ is the family of all bounded subsets in E(E′) then the corresponding $\mathfrak{S}$—topology on E′(E) will be called the strong topology and denoted by $s(E', E)$ $(s(E, E'))$.

DEFINITION 3. Arens topology $k(E', E)$ is defined by the $\mathfrak{S}$—topology where $\mathfrak{S}$ is a collection of balanced convex compact subsets of E.

DEFINITION 4. A subset K of E is said to be procompact on E if for every nbh V of zero in E there exists a finite subset $\{x_1, \cdots, x_n\}$ of E such that $K\subset\bigcup_{i=1}^n(x_i+V)$.

If $\mathfrak{S}$ is the collection of all precompact subsfts of E then the corresponding $\mathfrak{S}$—topology on E′ will be denoted by $\lambda(E', E)$.

DEFINITION 5. If $\mathfrak{S}$ is the collection of all balanced convex, $w(E, E')$—compact subsets of E′ then the corresponding $\mathfrak{S}$—topology on E′ will be called the Mackey topology and will be denoted by $m(E', E)$ on E′. Similarly we can define the Mackey topology $m(E, E')$ on E. In general, since a locally convex topology $\mathfrak{T}_E$ on E is compatible with the natural pairing of E and E′, the original topology $\mathfrak{T}_E$ on E is coarser than the Mackey topology $m(E, E')$.

DEFINITION 6. E is said to be relatively strong if the original topology $\mathfrak{T}_E$ on E coincides with the Mackey topology $m(E, E')$. [2, p. 505]

LEMMA 1. A locally convex Hausdorff E is relatively strong if either E is barrelled or E is metrizable. [1, prop. 36. 3]

LEMMA 2. If a locally convex space E is semimetrizable then E is relatively strong. [2, theorem 8. 3. 1]

LEMMA 3. Let E be an infrabarrelled with topology $\mathfrak{T}_E$ then E is relatively strong i.e., $\mathfrak{T}_E = m(E, E')$. [3, prop. 3. 6. 8]

LEMMA 4. A reflexive space is always barrelled. [3, cor. of prop. of 3. 8. 3]

From Lemma 1 and 4, a reflexive space is always relatively strong.

3. Theorems

We recall that E and F be two locally convex Hausdorff spaces and E′ and F′ be their duals throughout the following theorems.

Let $T: E \longrightarrow F$ be continuous linear map then given $y' \in F'$, the composition $y'oT: E \ni x \mapsto \langle T(x), y' \rangle \in K$ (the scalar field) is continuous linear form on E. This defines a mapping $T': F' \ni y' \mapsto y'oT \in E_0'$. The mapping T′ is called the transpose of T. It is obvicusly linear and satisfies $\langle x, T'(y') \rangle = \langle T(x), y' \rangle$.

Let A be a subset of E and B a subset of E′ then $T(A) \subset B$ implies $T'(B^\circ) = A^\circ$. Indeed, since $|\langle T(x), y' \rangle| = |\langle x, T'(y') \rangle| \leq 1$, we have $(T(A))^\circ = T'^{-1}(A^\circ)$. If $T(A) \subset B$ then $B^\circ \subset (T(A))^\circ = T'^{-1}(A^\circ)$. Thus $T(B^\circ) \subset A^\circ$.

THEOREM 1. If $T: E \longrightarrow F$ is continuous linear map with topologies $\mathfrak{T}_E$ and $\mathfrak{T}_F$ then it is continuous for the topologies $w(E, E')$ and $w(F, F')$.

Proof: Given nbh $U(y_1', \cdots, y_n', \varepsilon)$ of zero on F for $w(F, F')$, then the map $y'oT$ for $k=1, \cdots, n$ are continuous linear form on E for $\mathfrak{T}_E$. Hence we set for $k=1, 2, \cdots, n$, $x_k' = y'oT$ then $x_k' \in E'$ and we have, x is an element of a nbh $U(x_1', \cdots, x_n', \varepsilon)$ of zero for $w(E, E')$ implies

$$|\langle T(x), y_k' \rangle| = |\langle x, y_k'oT \rangle| = |\langle x, x_k' \rangle| \leqq \varepsilon \text{ for } k=1, 2, \cdots, n.$$

That is $T(U(x_1', \cdots x_n', \varepsilon)) \subset U(y_1', \cdots, y_n', \varepsilon)$. This means that the theorem is true.

THEOREM 2. If $T: E \longrightarrow F$ is continuous linear map with topologies $\mathfrak{T}_E$ and $\mathfrak{T}_F$, then its transpose $T': F' \longrightarrow E'$ is continuous linear map with topologies (a) $w(F', F)$ and $w(E', E)$, (b) $s(F', F)$ and $s(E', E)$, (c) $m(F', F)$ and $m(E', E)$, (d) $k(F', F)$ and $k(E, E)$, (e) $\lambda(F', F)$ and $\lambda(E', E)$.

Proof: (a) and (b). Given every nbh A° of zero in E′ for the topology $w(E', E)$ $(s(E', E))$ then A is finite (bounded) subset of E. Since T is continuous, $T(A)$ is also a finite (bounded) subset of

F and its polar $(T(A))^\circ$ is a nbh of zero in F for $w(F',F)$ $(s(F',F))$. Let $B=T(A)$ then we have $T'(B^\circ)\subset A^\circ$. Thus T' is continuous for the topologies $w(F',F)$ and $w(E',E)$ ($s(F',F)$ and $s(E,E)$).

(c) and (d). From theorem 1, since T is continuous for $w(E,E')$ and $w(F,F')$, if A is balanced convex $w(E,E')$—compact subset of E, then $B=T(A)$ is also a balanced convex $w(F,F')$—compact subset of F in the case (c). Let A be balanced convex $\mathfrak{T}_E$—compact subset of E then $B=T(A)$ is also a balanced convex $\mathfrak{T}_F$—compact subset of F in the case (d). Similarly as above (a) and (b), from $T'(B^\circ)\subset A^\circ$, T' is continuous for the topologies $m(F',F)$ and $m(E',E)$ in (c), $k(F',F)$ and $k(E',E)$ in (d).

(e). Let A be a precompact subset of E and V a nbh of zero in F, then, since T is continuous, $T^{-1}(V)$ is also a nbh of zero in E. By Definition 4, there exists a finite sucset $\{x_1,\cdots,x_n\}$ of E such that $A\subset \cup_{i=1}^{n}(x_i+T^{-1}(V))$. From linearity of T, we have $B=T(A)\subset \cup_n^{i=1}(T(x_i)+V)$. Thus $T(A)$ is a precompact subset of F. $T'(B^\circ)\subset A^\circ$ means that T' is continuous for the topology $\lambda(F',F)$ and $\lambda(E',E)$. Q.E.D.

If $T: E\longrightarrow F$ is continuous linear map for $w(E,E')$ and $w(F,F')$, then $T': F'\longrightarrow E'$ is continuous linear map for $w(F',F)$ and $w(E',E)$ because of theorem 2 (a). It follows from

$$T''=(T')': E\ni x\mapsto xoT'\in F \text{ and } \langle T(x),\ y'\rangle=\langle x, T'(y')\rangle \text{ that } T''=T.$$

THEOREM 3. If $T': F'\longrightarrow E'$ is coutingous linear map for $w(F',F)$ and $w(E',E)$ then T' is also $s(F',F)$—$s(E',E)$ continuous.

Proof: From theorem 2(a), if $T': F'\longrightarrow E'$ is continuous for $w(F',F)$ and $w(E',E)$ then $T''=T: E\longrightarrow F$ is $w(E,E')$—$w(F,F')$ continuous. Thus from theorem 2(b), $T': F'\longrightarrow E'$ is $s(F',F)$—$s(E',E)$ continuous.

Remark: If $T': F'\longrightarrow E'$ is continuous for $s(F',F)$ and $s(E',E)$, then it is not necessarily continuous for $w(F',F)$ and $w(E',E)$ [3, p. 258]. We have, however, if F is semireflexive, then the converse of theorem 3 is true. [3, prop. 3.12.6]

In general, if a map $T: E\longrightarrow F$ is continuous for $w(E,E')$ and $w(F,F')$ then it is not necessarily continuous for the original topologies $\mathfrak{T}_E$ and $\mathfrak{T}_F$. [3, p. 257] We have, however, the following result.

THEOREM 4, If $T: E\longrightarrow F$ is continuous linear map for $w(E,E')$ and $w(F,F')$, then T is continuous for $m(E,E')$ and $m(F,F')$.

Proof: Let U be a nbh of zero on F for $m(F,F')$ then there exists a balanced convex $w(F',F)$—compact subset of B of F' such that $B^\circ=U$. Since, by theorem 2(a), $T': F'\longrightarrow E'$ is continuous for $w(F',F)$ and $w(E',E)$, $A=T'(B)$ is also a balanced convex $w(E',E)$—compact subset of E for $m(E',E)$. Hence $A^\circ=V$ is a nbh of zero in E for $m(E,E')$. We have $T''(A^\circ)=T(A^\circ)\subset B^\circ$. That is $T(V)\subset U$ which proves that T is continuous for $m(E,E')$ and $m(F,F')$.

THEOREM 5. If E is relatively strong then $T: E\longrightarrow F$ is continuous linear map for $w(E,E')$ and $w(F,F')$ implies it is also continuous for the original topologies $\mathfrak{T}_E$ and $\mathfrak{T}_F$.

Proof: From theorem 4, $T: E\longrightarrow F$ is $m(E,E')$—$m(F,F')$ continuous. Since $m(F,F')$ is finer than $\mathfrak{T}_F$, $T: E\longrightarrow F$ is continuous for $\mathfrak{T}_E=m(E,E')$ and $\mathfrak{T}_F$.

THEOREM 6. $T: E\longrightarrow F$ is continuous linear map for $m(E,E')$ and $w(F,F')$ if and only if T is continuous linear map for $m(E,E')$ and $m(F,F')$.

Proof: From theorem 4 and 5, the theorem is true.

THEOREM 7. If E is (a) barrelled (b) metrizable (c) semimetrizable (E is not necessarily Hausdorff) (d) infrabarrelled (e) reflexive then $T : E \longrightarrow F$ is weakly continuous linear map implies it is continuous for the original topologies $\mathfrak{T}_E$ and $\mathfrak{T}_F$, in each case.

Proof: From Lemma 1, 2, 3, 4, and theorem 5, the theorem 7 is true.

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