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ON THE SUBALGEBRA $H^\infty[B]$ IN $L^\infty(R)$

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1. Introduction

R.G. Douglas [2] has asked whether every closed subalgebra of L^∞ containing H^∞ is generated by H^∞ and the inverses of the functions in H^∞ that are invertible in the algebra. For this it is natural to think the bounded analytic functions in the upper half plane π^+ by applying a Cayley transformation. Thus we shall consider the algebras of functions on π^+.

Let $L^\infty(R)$ and $H^\infty(R)$ be corresponding functions on π^+ of L^∞ and H^∞ on the unit circle, respectively. Then those in $H^\infty(R)$ are the boundary functions for bounded analytic functions in π^+. In this sense we shall consistently identify the functions in $H^\infty(R)$ with their natural analytic extensions into π^+, i.e., the functions in $H^+(\pi^+)$.

Let B denote the set of the bounded uniformly continuous functions on R. We define $H^\infty[B]$ to be the closed subalgebra of $L^\infty(R)$ generated by $H^\infty(R)$ and B.

2. The closed subalgebra $H^\infty[B]$ in $L^\infty(R)$

For $\alpha \geqq 0$, let ϕ_α denote the function $\exp(i\alpha x)$. Then the functions ϕ_α belong to $H^\infty(R)$. Our aim in this section is to show that $H^\infty[B]$ is the closure in $L^\infty(R)$ of the subalgebra $\bigcup_{\alpha>0}\bar{\phi}_\alpha H^\infty(R)$, where $\bar{\phi}_\alpha$ is the conjugate of ϕ_α.

Let us first consider an entire function of exponential type α. Suppose that f is in B, and put

$$K(t)=\frac{2}{\pi}[t^{-1}\sin^t/2]^2.$$

Then $\int_{-\infty}^{\infty}K(t)\,dt=1$

and the function

$$G(x)=\int_{-\infty}^{\infty}K(t)f\left(z+\frac{t}{\alpha}\right)dt$$

is an entire function of exponential type α. [1, p. 249]

THEOREM 1: $H^\infty[B]$ is the closure in $L^\infty(R)$ of the subalgebra $\bigcup_{\alpha>0}\bar{\phi}_\alpha H^\infty(R)$.

Proof. For $\alpha>0$ the subspace $\bar{\phi}_\alpha H^\infty(R)$ is contained in $H^\infty[B]$. The subspace $\bigcup_{\alpha>0}\bar{\phi}_\alpha H(R)$ is an algebra and its closure is also an algebra. Since $H^\infty[B]$ is closed, it must contain the closure of the algebra $\bigcup\bar{\phi}_\alpha H^\infty(R)$.

To establish that $H^\infty[B]$ is contained in the closure of the algebra $\bigcup_{\alpha>0}\bar{\phi}_\alpha H^\infty(R)$, assume first that f is a function in $H^\infty(R)$.

Then since

$$|f-e^{-i\alpha x}f|\leqq 2\ ||f||_\infty |\sin\frac{\alpha x}{2}|\to 0 \text{ as } \alpha\to 0$$

we have

$$\text{dist }(f, e^{-i\alpha x}H^\infty(R)\to 0 \text{ as } \alpha\to 0.$$

Thus f is in the closure of the algebra $\underset{\alpha>0}{U}\bar{\phi}_\alpha H^\infty(R)$.

To complete the proof of theorem 1, assume that f is a function in B. Let$g=\phi_\alpha f$. Then g is in B, and the function

$$G(Z)=\int_{-\infty}^{\infty}K(t)g\left(z+\frac{t}{\alpha}\right)dt$$

is an entire function of exponential type α, where

$$K(t)=\frac{2}{\pi}\left[\frac{\sin t/2}{t}\right]^2.$$

Since g is uniformly continuous on R,

$$\begin{aligned}&|g(X)|-G(X)|\\ &=|\int_{-\infty}^{\infty}K(t)\left[g\left(X+\frac{t}{\alpha}\right)-g(X)\right]dt|\\ &\leqq\sup_x|g\left(X+\frac{t}{\alpha}\right)--g(X)|\to 0 \text{ as } \alpha\to\infty.\end{aligned}$$

Thus $\qquad |f-e^{-i\alpha x}G|\to 0$ as $\alpha\to\infty$

and since G is in $H^\infty(R)$,

$$\text{dist }(f,\bar{\phi}_\alpha H^\infty)\to 0 \text{ as } \alpha\to\infty.$$

Hence f is in the closure of the algebra $\underset{\alpha>0}{U}\bar{\phi}_\alpha H^\infty$.

3. The multiplicative Poisson integral in $H^\infty(B)$

For f in $L^\infty(R)$ and $z\in\pi^+$, we let $f(z)$ stand for the value at z of the Poisson integral of f. Thus

$$f(Z)=\frac{1}{\pi}\int_{-\infty}^{\infty}P_y(X-t)f(t)dt,\ Z=x+iy$$

where $P_y(x-t)$ is the Poisson kernel:

$$P_y(X-t)=\frac{v^2}{(X-t)^2+y^2}$$

For $a>0$ we define

$$f_a(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}P_a(X-t)f(t)dt.$$

LEMMA 1: If f is in B and g is in $L^\infty(R)$, then

$$||f_a g_a-(fg)_a||_\infty\to 0 \text{ as } a\to 0.$$

Proof. we note that, for x+ia in the upper half plane,

$$\begin{aligned}&f_a(x)g_a(x)-(fg)_a(x)\\ &=[f_a(x)-f(x)]g_a(x)+[f(x)g_a(x)-(fg)_a(x)]\\ &=A_a(x)+B_a(x).\end{aligned}$$

Since f is in $L^\infty(R)$,

$$\lim_{a\to 0} \sup\{|A_a(x)| : X+ia\in\pi^+\} = 0 \text{ by the theorem. } [3, \text{ p. } 123]$$

Thus we have to show that

$$\lim_{a\to 0} \sup\{|B_a(x)| : x+ia\in\pi^+\} = 0$$

and hence this result yields the lemma.

Using the representations of g_a and $(fg)_a$, we have

$$B_a(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}(f(x)-f(t))g(t)P_a(x-t)\,dt$$
$$= \frac{1}{\pi}\int_{|x-t|<\delta}(f(x)-f(t))g(t)P_a(x-t)\,dt$$
$$+ \frac{1}{\pi}\int_{|x-t|\geqq\delta}(f(x)-f(t))g(t)P_a(x-t)\,dt.$$

Hence

$$|B_a(x)| \leqq \|g\|_\infty \sup_{|x-t|<\delta}|f(x)-f(t)| + 2\|f\|_\infty \|g\|_\infty \sup_{|x-t|\geqq\delta} P_a(x-t)$$

The supremum in the first term on the right side tends to 0, since f is uniformly continuous on R, and since

$$\lim_{a\to 0} \sup_{|x-t|\geqq\delta} P_a(x-t) = 0.$$

We see that

$$\lim_{a\to 0}|B_a(x)| = 0.$$

THEORM 2: If f and g are in $H^\infty[B]$, then

$$\|f_a g_a - (fg)_a\|_\infty \to 0 \text{ as } a\to 0.$$

Proof. Since the algebra $\underset{\alpha>0}{U}\bar{\phi}_\alpha H^\infty(R)$ is dense in $H^\infty[B]$ by theorem 1, it will suffice to prove theorm 2 for the functions f and g in $\underset{\alpha>0}{U}\bar{\phi}_\alpha H^\infty(R)$. Let $f=\bar{\phi}_\alpha f_1$, and $g=\bar{\phi}_\beta g_1$ be two functions in $\underset{\alpha>0}{U}\bar{\phi}_\alpha H^\infty(R)$, where f_1 and g_1 are in $H^\infty(R)$. Since the Poisson integral is multiplicative on $H^\infty(R)$, the difference $f_a g_a - (fg)_a$ can be broken up as follows:

$$\begin{aligned}
& f_a g_a - (fg)_a \\
&= (\bar{\phi}_\alpha f_1)_a(\bar{\phi}_\beta g_1)_a - (\bar{\phi}_\alpha\bar{\phi}_\beta f_1 g_1)_a \\
&= [(\bar{\phi}_\alpha f_1)_a - (\bar{\phi}_\alpha)_a(f_1)_a](\bar{\phi}_\beta g_1)_a \\
&+ (\bar{\phi}_\alpha)_a(f_1)_a[(\bar{\phi}_\beta g_1)_a - (\bar{\phi}_\beta)_a(g_1)_a] \\
&+ (\bar{\phi}_\alpha)_a[(\bar{\phi}_\beta)_a(f_1 g_1)_a - (\bar{\phi}_\beta f_1 g_1)_a] \\
&+ [(\bar{\phi}_\alpha)_a(\bar{\phi}_\beta f_1 g_1)_a - (\bar{\phi}_\alpha\bar{\phi}_\beta f_1 g_1)_a].
\end{aligned}$$

An application of lemma to each term on the right side shows that.

$$\|f_a g_a - (fg)_a\|_\infty \to 0 \text{ as } a\to 0.$$

COROLLARY: If f is an invertible function in $H^\infty[B]$, then its Poisson integral is bounded away from 0 in some strip $0<I_m z<\varepsilon$.

Proof: Assume that f is invertible in $H^\infty[B]$, and let $g=f^{-1}$. Then, by the theorm 2, the product of the Poisson integral of f with the Poisson integral of g must be uniformly close to 1 as $a\to 0$. If we choose ε so that

$$\|f_a g_a - 1\|_\infty < \frac{1}{2} \text{ for } 0 < a < \varepsilon,$$

we have

$$|f_a(x)| > \frac{1}{2\|g\|_\infty} \text{ for } 0 < a < \varepsilon$$

so f_a satisfies the condition.

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