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Consistency of the Cramér-von Mises Test

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1. Introduction and summary

It is valuable to study the properties of a test for goodness of fit. The oldest and widely known goodness of fit test is chi-square test for goodness of fit which was introduced by Pearson. In this paper we shall study the properties of the Cramér-von Mises test.

Let $X_1, X_2, \cdots\cdots, X_n$ be a random sample of n observations from some unknown distribution function $F(x)$. Let $X_{(1)} < X_{(2)} < \cdots\cdots < X_{(n)}$ are the ordered observations. And define the empirical distribution function as following.

$$
\begin{aligned}
F_n(x) &= 0 && \text{if } x < X_{(1)} \\
&= \frac{k}{n} && \text{if } X_{(k)} \le x < X_{(k+1)} \\
&= 1 && \text{it } x \ge X_{(n)}
\end{aligned}
$$

In other word, $nF_n(x)$ is equal to the number of observations in the sample that are smaller than or equal to x. If $F_0(x)$ is a completely specified continuous distribution function, we may reject the null hypothesis that $F_0 \equiv F$ for large value of

$$d_1 = \frac{1}{12n} + \sum_{k=1}^{n} \left[F_0(x_{(k)}) - \frac{2k-1}{2n} \right]^2$$

i. e. rejects the null hypothesis at the level of significance α if the d_1 exceeds the $1-\alpha$ quantile $w_{1-\alpha}$ as given by Anderson and Darling [1]. We say that d_1 is one sample Cramér-von Mises test statistic. Let $F_n(x)$ be the empirical distribution function based on the random sample $X_1, X_2, \cdots\cdots, X_n$ with unknown distribution function $F(x)$ and let $G_m(x)$ be the empirical distribution function based on the other random sample $Y_1, \cdots\cdots, Y_m$ with unknown distribution function $G(x)$. We say that d_2 is two sample Cramér-von Mises statistic

$$d_2 = \frac{\text{[illegible]}}{(m+n)^2} \sum_{x=x_i,\ x=x'_j} [F_n(x) - G_m(x)]^2 \qquad [2]$$

Reject the null hypothesis $F(x) \equiv G(x)$ at the approximate level α if d_2 exceeds the $1-\alpha$ quantile $w_{1-\alpha}$ as given by Anderson. and Darling, [1].

Thompson, R. [4] showed that the one sample Cramér-von Mises test is biased.

The objects of this paper are as follows;

In Theorem 2-1, we show that the one sample Cramér-von Mises test is consistent.

In Theorem 2-2, We show that the two sample Cramér-von Mises test is consistent.

2. Consistency

Definition. A test is said to be consistent against a class of alternatives if with increasing sample size probabllity that it rejects the hypothesis being tested tends 1 whenever one of the alternatives in the class is true.

The following lemmas are discussed in [3].

Lamma 1. Let $F_n(x)$ be the empirical distribution of a sample size n from $F(x)$, then

$$P\left[F_n(x)=\frac{k}{n}\right]=\binom{n}{k}[F(x)]^k\ [1-F(x)]^{n-k},\quad k=0,1,2,\cdots\cdots,n$$

According to lemma 1,

Lamma 2. Let $\mathcal{E}[F_n(x)]$, $var[F_n(x)]$ denote the mean and variance of $F_n(x)$, respectively; then

$$\mathcal{E}[F_n(x)]=F(x)$$

$$var[F_n(x)]=\frac{1}{n}F(x)[1-F(x)]$$

Lemma 2 show that for fixed x, $F_n(x)$ is an unbiased and mean-squared-error consistent estimator of $F(x)$. And by the following Glivenko-Cantelli theorem.

$$P[\sup_x|F_n(x)-F(x)|\longrightarrow 0]=1 \qquad \text{as } n\longrightarrow\infty$$

the estimating function $F_n(x)$ of the $F(x)$ converges to $F(x)$ uniformly for all x with probability one. Thus the $F_n(x)$ is a consistent estimator of $F(x)$.

Theorem 2-1. One sample Cramér-von Mises test is consistent.

Proof. Let the null hypothesis be $F_0(x)=F(x)$ and the alternative hypothesis be $F_1(x)$, where $F_1(x)\neq F_0(x)$. Let $\Delta=F_0(x_{(k)})-F_1(x_{(k)})$, that is, $F_0(x_{(k)})=F_1(x_{(k)})+\Delta$. Then Δ is a number which is concerned with sample size n such that $0<|\Delta|\leq 1$. However, we choose then sufficiently large, Δ doesn't converge to 0 since F_0 is different from F_1. Let

$$d_1=\frac{1}{12n}+\sum_{k=1}^{n}\left[F_0(x_{(k)})-\frac{2k-1}{2n}\right]^2$$

$$d_1'=\frac{1}{12n}+\sum_{k=1}^{n}\left[F_1(x_{(k)})-\frac{2k-1}{2n}\right]^2$$

Then the probability of rejecting the null hypothesis $P\{d_1>w_{1-\alpha}\}$ when a alternative hypothesis is true is

$$\begin{aligned}P\{d_1>w_{1-\alpha}\}&=1-P\{d_1\leq w_{1-\alpha}\}\\&=1-P\left\{\frac{1}{12n}+\sum_{k=1}^{n}\left[F_0(x_{(k)})-\frac{2k-1}{2n}\right]^2\leq w_{1-\alpha}\right\}\\&=1-P\left\{\frac{1}{12n}+\sum_{k=1}^{n}\left[F_1(x_{(k)})+\Delta-\frac{2k-1}{2n}\right]^2\leq w_{1-\alpha}\right\}\\&=1-P\left\{d_1'+2\Delta\sum_{k=1}^{n}F_1(x_{(k)})-\Delta n+\Delta^2 n\leq w_{1-\alpha}\right\}\end{aligned}$$

Here, we have

$\sum_{k=1}^{n}F_1(x_{(k)})\longrightarrow\frac{n+1}{2}$ with probability one. Thus the equation (1) tends

$$1-P\{d_1'+\Delta^2 n\leq w_{1-\alpha}\}$$

$$=1-P\left\{-\frac{\delta}{\sqrt{n}}\leq\Delta\leq\frac{\delta}{\sqrt{n}}\right\},\ \text{where } \delta^2=\max(0, w_{1-\alpha}-\Delta-d_1')$$

$$=1-P\left\{-\frac{\delta}{\sqrt{n}}\leq F_0(x_{(k)})-F(x_{(k)})\leq\frac{\delta}{\sqrt{n}}\right\}$$

$$=1-P\left\{\frac{\kappa-\delta}{\eta}\leq\frac{k}{\eta}\leq\frac{\kappa+\delta}{\eta}\right\}$$

$$=1-\int_{\lambda_1}^{\lambda_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt \qquad \text{by } F_1(x)=F(x) \text{ and lemma 2.}$$

where $\kappa=[F_n(x_{(k)})-F_0(x_{(k)})]\sqrt{n}$, $\eta=\sqrt{F_1(x_{(k)})[1-F_1(x_{(k)})]}$,

$\lambda_1=\frac{\kappa-\delta}{\eta}$ and $\lambda_2=\frac{\kappa+\delta}{\eta}$

For sufficienthy large n, both $\kappa-\delta$ and $\kappa+\delta$ have the same sign. Thus the probability that it rejects the hypothesis when a alternative is true tends to 1. Q.E.D.

Lemma 3. $\mathrm{var}[F_n(x)-G_n(y)]=\frac{1}{n}[F(x)-G(y)]\ [1-F(x)+G(y)]$

Theorem 2-2. Two sample Cramér-von Mises test is consistent.

Proof. Two sample Cramér-von Mises test statistic d_2 is defined as

$$d_2=\frac{mn}{(m+n)^2}\sum_{\substack{x=x_i\\x=y_j}}[F_n(x)-G_m(x)]^2$$

Then,

$$d_2=\frac{mn}{(m+n)^2}\left\{\sum_{i=1}^{n}[F_n(x_i)-G_m(x_i)]^2+\sum_{j=1}^{m}[F_n(y_j)-G_m(y_j)]^2\right\}$$

According to the Glivenko-Cantelli theorem, we have

$$d_2=\frac{mn}{(m+n)^2}\left\{\sum_{i=1}^{n}[F_n(x_i)-F_0(x_i)]^2+\sum_{j=1}^{m}[F_n(y_j)-F_0(y_j)]^2\right\}$$

$$=\frac{mn}{(m+n)^2}\left\{\sum_{i=1}^{n}\left[F_0(x_i)-\frac{i}{n}\right]^2+\sum_{j=1}^{m}\left[F_0(y_j)-\frac{j}{n}\right]^2\right\}$$

Let $\Delta=F_0(x_i)-F_1(x_i)$, then the above equation approaches

$d_2'+\frac{mn\Delta^2}{m+n}+\frac{m+1}{2}\left(1-\frac{m}{n}\right)$ where

$$d_2'=\frac{mn}{(m+n)^2}\left\{\sum_{i=1}^{n}\left[F_1(x_i)-\frac{i}{n}\right]^2+\sum_{j=1}^{m}\left[F_1(y_j)-\frac{j}{n}\right]^2\right\}. \text{ Thus}$$

$P\{d_2>w_{1-\alpha}\}=1-P\{d_2\leq w_{1-\alpha}\}$

$$=1-P\left\{d_2'+\frac{mn}{m+n}\Delta^2+\frac{m+1}{2}\left(1-\frac{m}{n}\right)\leq w_{1-\alpha}\right\}$$

$$=1-P\{-\eta\leq\Delta\leq\eta\},\ \text{where } \eta^2=\frac{\max\left(0, w_{1-\alpha}-d_2'-\frac{m+1}{2}\left(1-\frac{m}{n}\right)\right)}{\frac{mn}{m+n}}$$

$$=1-P\left\{\frac{\kappa-\eta}{\sigma}\leq\frac{F_n(x_i)-G_m(y_j)-[F_1(x_i)-G(y_j)]}{\sigma}\leq\frac{\kappa+\eta}{\sigma}\right\}$$

$$=1-\int_{\lambda_1}^{\lambda_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$$

where $\kappa=F_n(x_i)-G_m(y_j)-[F_0(x_i)-G(y_j)]$,

$$\sigma=\sqrt{\frac{m+n}{mn}}\sqrt{[F(x)-G(y)][1-F(x)+G(y)]},$$

$$\lambda_1=\frac{\kappa-\eta}{\sigma} \quad \text{and} \quad \lambda_2=\frac{\kappa+\eta}{\sigma}$$

Since λ_1 and λ_2 are same sign as $m, n \longrightarrow \infty$, $\int_{\lambda_1}^{\lambda_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$ tendsto 0 with probability 1. Q.E.D.

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