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A Note on c-semidevelopable spaces and c-first countable spaces

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By Robert W. Heath (1) and Charles C. Alexander (2), a space X is semidevelopable if and only if X is first countable and semistratifiable, and by Harold W. Martin (3), the c-semistratifiable space was introduced.

Thus the following diagram is described:

$$\text{semidevelopable} \longleftrightarrow \begin{cases} \text{semistratifiable} \longrightarrow \text{c-semistratifiable} \\ \text{first countable} \end{cases}$$

In this paper we will define c-semidevelopable space and c-first countable space, and then we will be shown that the following diagram holds:

$$\text{semidevelopable} \begin{array}{l} \nearrow \text{semistratifiable} \\ \quad \longrightarrow \\ \searrow \text{first countable} \end{array} \begin{array}{c} \longrightarrow \\ \text{C-SEMIDEVELOPABLE} \\ \longrightarrow \end{array} \begin{array}{l} \nearrow \text{c-semistratifiable} \\ \quad \longrightarrow \\ \searrow \text{C-FIRST COUNTABLE} \nearrow \end{array} \searrow \text{point is } G\delta$$

1. Definitions and characterizations.

We adopt the convention that F is a closed set and K is a closed and compact set in this paper.

Definition 1. For a space X, $g: N \times X \longrightarrow \mathfrak{F}$ is a countably open cover(COC) function provided that $x \in g(n, x)$ for each $x \in X$ and n, and $g(n+1, x) \subset g(n, x)$.

Definition 2. A space X is semidevelopable iff there is a sequence $\{\gamma_1, \gamma_2, \cdots\cdots\}$ of covers of X such that $\{st(x, \gamma_n);\ n=1, 2, \cdots\cdots\}$ forms a local system of neighborhood at x for each $x \in X$.
Equivalently,
there exists a sequence $\{\gamma_1, \gamma_2 \cdots\cdots\}$ of cover of X such that $x \in st(x, \gamma_n)^\circ$ for each $x \in X$ and if $x \in U$ for some open set U then there exists n such that $st(x, \gamma_n) \subset U$.

Definition 3. A space X is semistratifiable iff there is a sequence $\{g_n\}_{n=1}^{\infty}$ of function from X into the collection of open sets of X such that (1) $\bigcap_{n=1}^{\infty} g(n, x) = cl\{x\}$ for each x, and (2) if y is a point of X and $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in X, with $y \in g(n, x_n)$ for all n, then $\{x_n\}_{n=1}^{\infty}$ converges to y. (This definition is the characterized from by Geoffrey D. Creede.)
Equivalently,
$F = \bigcap_{n=1}^{\infty} g(n, F)$ and if $x \in X - F$ then there exists n such that $x \notin g(n, F)$.

Definition 4. A space X is first countable iff $F \cap \{x\} = \phi$ then there exists n such that $F \cap g(n, x) = \phi$.
Equivalently,

if $x \in U$ for some open set there exists n such that $g(n, x) \subset U$. (Or, if $x_n \in g(n, x)$ then $\{x_n\}$ converges to x.)

Definition 5. A space X is c-semistratifiable if there is a sequence $\{g(n, x); x \in X, n=1, 2, \cdots\cdots\}$ of open subsets of X which satisfies the following conditions:

(1) $x \in g(n, x)$

(2) $g(n+1, x) \subset g(n, x)$

(3) If A is a closed compact subset of X and $x \in X-A$, then there exists n such that $x \notin g(n, a)$ for each $a \in A$.

Equivalently,

$K = \bigcap_{n=1}^{\infty} g(n, k)$ and if $\{x\} \cap K = \phi$ then there exists n such that $x \cap g(n, K) = \phi$.

Definition 6. A space X is c-semidevelopable iff there is a sequence $\{\gamma_1, \gamma_2, \cdots\cdots\}$ of cover of X such that $x \in st(x, \gamma_n)^{\circ}$ for each $x \in X$ and if $x \in U$ for some cocompact open set U then there exists n such that $st(x, \gamma_n) \subset U$.

Equivalently,

there is a sequence $\{\gamma_1, \gamma_2, \cdots\cdots\}$ of covers of X such that $x \in st(x, \gamma_n)^{\circ}$ for each $x \in X$ and if $x \notin K$ then there exists n such that $K \cap st(x, \gamma_n) = \phi$.

Definition 7. A space X is c-first countable iff $K \cap \{x\} = \phi$ then there exists n such that $K \cap g(n, x) = \phi$.

Equivalently,

if V is the complement of closed and compact set and $x \in V$, then there exists n such that $g(n, x) \subset V$.

Definition 8. A space X is a q-space iff $x_n \in g(n, x)$ for $n=1, 2, \cdots\cdots$, then the sequence $\{x_n\}$ has a cluster point.

2. The relations between the spaces in diagram.

Theorem 1. If X is a c-semidevelopable space then X is a c-first countable space.

proof. Let $g(1, x) = st(x, \gamma_1)^{\circ}$, $g(2, x) = st(x, \gamma_1)^{\circ} \cap st(x, \gamma_2)^{\circ}, \cdots\cdots$, $g(n, x) = st(x, \gamma_1)^{\circ} \cap \cdots\cdots \cap st(x, \gamma_n)^{\circ}$. Then g is a COC function and $g(n, x) \subset st(x, \gamma_n)$. Also, if $K \cap \{x\} = \phi$ for any K, then there exists n such that $\phi = K \cap st(x, \gamma_n) \supset K \cap g(n, x)$. Therefore X is a c-first countable.

Theorem 2. If X is a c-semdevelopable space, then X is a c-semistratifiable space.

proof. Take a function g as theorem 1. If $\{x\} \cap K = \phi$ for any K, then there exists n such that $\{x\} \cap st(K, \gamma_n) = \phi$. Because, $x \in st(K, \gamma_n)$; iff $\exists y \in K$ such that $x \in st(y, \gamma_n)$; iff $\exists y \in K$ such that $y \in st(x, \gamma_n)$ iff $K \cap st(x, \gamma_n) \neq \phi$. This contradicts to the hypothesis. Hence $\{x\} \cap st(K, \gamma_n) = \phi$. But $\{x\} \cap g(n, K) \subset \{x\} \cap st(K, \gamma_n) = \phi$. Therefore X is a c-semistratifiable.

Theorem 3. If X is a c-first countable and c-smistratifiable space, then X is a c-semidevelopable space.

proof. Let f be a COC function such that if $K \not\ni x$ then there exists n such that $K \cap f(n, x) = \phi$. And let g be a COC function such that if $x \notin K$ then there exists n such that $x \notin g(n, K)$.

Define $l(n, x) = g(n, x) \cap f(n, x)$ and take $\gamma_1 = \{\{x, y\} \mid x \in l(n, y) \text{ or } y \in l(n, x)\}$. Then $st(x, \gamma_n) = \{y \mid x \in l(n, y) \text{ or } y \in l(n, x)\} = \{y \mid x \in l(n, y)\} \cup \{y \mid \in l(n, x) = l(n, x) \cup y \mid x \in l(n, y)\}$. Therefore

$st(x, \gamma_n)^\circ \supset l(n, x) \ni x$, and if $x \notin K$ then there exists n such that $K \cap st(x, \gamma_n) = \phi$.

Theorem 4. Any point is G_δ set iff there exists a COC function $g : N \longrightarrow \mathfrak{F}$ such that if $x \neq y$ then there exists n such that $y \notin g(n, x)$.

proof. necessity. Let G_n be a sequence of open sets such that $\{x\} = \bigcap_{n=1}^\infty G_n$ for each x. If we take $G_n = g(n, x)$, then $\bigcap_{n=1}^\infty g(n, x) = \{x\} = \bigcap_{n=1}^\infty G_n$.

Sufficiency. Let $G_1 = g(1, x)$, $G_1 \cap G_2 = g(2, x)$, ……, $G_1 \cap \cdots\cdots \cap G_n = g(n, x)$, ……,then $g: N \times X \longrightarrow \mathfrak{F}$ is a COC function and since $\{x\} = \bigcap_{n=1}^\infty G_n = \bigcap_{n=1}^\infty g(n, x)$, there exists n such that if $x \neq y$ then $y \notin g(n, x)$.

Theorem 5. In a regular q-space, the followings are equivalent:

1) first countable
2) $x_n \in g(n, x) \Longrightarrow cl\{x_n\} \subset \{x\}$
3) c-first countable
4) point is G_δ

proof. 1)$\Rightarrow$2) : Clear.

2)$\Rightarrow$3) : Let $x_n \in g(n, x)$ and $x \in U$ where U is a cocompact open set. Assume $\{i \mid x_i \notin U\}$ is infinite; $\{x_i \mid x_i \in X - U\}$ is infinite. Since $X - U$ is a compact set, $\{x_i \mid x_i \in X - U\}$ has a cluster point z which is different from x in $X - U$. This contradicts to $cl\{x_n\} \subset x$. Hence there exists n such that $g(n, x) \subset U$.

3)$\Rightarrow$4) : Since y is a closed and compact set and $y \neq x$, there exists n such that $\{y\} \cap g(n, x) = \phi$.

4)$\Rightarrow$1) By D.J. Lutzer (4).

3. Examples.

Let X be R^2 with a topology as follows: $x \neq 0$ (0 is origin), $\{g(n, x) = S\left(x, \frac{1}{n}\right) \mid n \in N\}$ is a local base at x and U is a neighborhood of 0 iff $U \cap l_\theta$ contains an open interval containing 0, where l_θ = a straight line through 0 intersecting x-axis with angle θ for any θ.

Then X is a c-first countable but not first countable.

References

1. Robert W. Heath, Arc-wise connectedeness in semimetric spaces, Pacific J. Math. vol 12(1962), 1301-1319.
2. Charles C. Alexander, Semidevelopable space and quotient images of metricspaces, Pacific J. Math. vol 37(1971), 277-293.
3. Harold W.Martin, Metrizability of M-spaces, Canad. J. Math. vol 25(1973) 840-841.
4. D.J. Lutzer, Semimetrizable and stratifiable spaces, General Topology and Appl, 1(1971), 43-48.
5. R.E. Hodel, Spaces defined by sequences of open covers which guarentee that certain sequences have cluster points, Duke Math. J. vol 39(1972), 253-263.
6. Geoffrey D. Creede, Concerntng semistratifiable spaces, Pacific J. Math. vol 32 (1970), 47-54.
7. C. Borges, On stratifiable spaces, Pacific J. Math. vol 17(1966), 1—16.
8. Michal Henry, Stratifiable spaces, semistratifiable spaces and thier relation through mappings, Pacific J. Math vol 37(1971), 697-700.
9. J. Kelley, General Topollgy, Van Nostrand, 1955.