韓國數學敎育學會誌〈數學敎育〉
1976. 12. Vol. XV No.1

Journal of the Korea Society of Mathematical Education
Dec. 1976. Vol. XV No. 1

# On the Characteristic Orthogonal Nonholonomic Frames in Vn

by

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## 1. INTRODUCTION

Introducing a set of 4 linearly independent basic nulll vectors V. Hlavaty ([3]) introduced the concept of the nonholonomic frames and used it successfully as a tool to develop the algebra of the unified field theory in the space-time $X_4$. In our previous paper ([2]) we introduced the concept of the general nonholonomic frames and orthogonal nonholonomic frames to an n-dimensional Riemannian space Vn and investigated their elementary properties.

This paper is a direct continuation of [2]. The purpose of the present paper is to construct the characteristic orthogonal nonholonomic frame of Vn determined by a symmetric tensor $a_{\lambda\mu}$, composed of n different eigenvectors of $a_{\lambda\mu}$, and to derive its particular properties.

## 2. PRELIMINARY RESULTS

In this section, results obtained in our previous paper [2], which are necessary for our further discussions, will be introduced without proof.

Let $V_n$ be a n-dimensional Riemannian space referred to a real coordinate system $x^\nu$ and defined by a fundamental metric tensor $h_{\lambda\mu}$, whose determinant

$$(2.1)\qquad h \overset{\text{def}}{=} \mathrm{Det}((h_{\lambda\mu}))\neq 0.$$

According to (2.1) there is a unique tensor $h^{\lambda\nu}=h^{\nu\lambda}$ defined by

$$(2.2)\qquad h_{\lambda\mu}h^{\lambda\nu} \overset{\text{def}}{=} \delta_\mu^\nu.$$

The tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ wll serve for raising and lowering indices of tensor quantities in $V_n$ in the usual manner.

If $\underset{i}{e}^\nu$, $(i=1,\cdots,n)$, are a set of n linearly independent unit vectors, then there is a unique reciprocal set of n linearly independent covariant vectors $\overset{i}{e}_\lambda$, $(i=1,\cdots,n)$, satisfying

$$(2.3)\qquad \underset{i}{e}^\nu \overset{i}{e}_\lambda=\delta_\lambda^\nu,\quad \underset{j}{e}^\lambda \overset{i}{e}_\lambda=\delta^i_j.$$

With the vectors $\underset{i}{e}^\nu$ and $\overset{i}{e}_\lambda$ a *nonholonomic frame* of $V_n$ is defined in the following way: If $T^{\nu\cdots}_{\lambda\cdots}$

are holonomic components of a tensor density of weight p, then its *nonholonomic components* are defined by

(2.4)a $$T^{i\cdots}_{j\cdots} \overset{\text{def}}{=} A^{-p}\ T^{\nu\cdots}_{\lambda\cdots}\overset{i}{e}_{\nu}\underset{j}{e}^{\lambda}\cdots{}^{(*)},\quad A=\mathrm{Det}(\!(\overset{i}{e}_{\lambda})\!).$$

An easy inspection of (2.3) and (2.4) a shows that

(2.4)b $$T^{\nu\cdots}_{\lambda\cdots}=A^{p}T^{i\cdots}_{j\cdots}\underset{i}{e}^{\nu}\overset{j}{e}_{\lambda}.$$

The nonholonomic frame in $V_n$ constructed by the unit vectors $\underset{i}{e}^{\nu}$, $(i=1,\cdots,n)$, tangent to the n congruences of an orthogonal ennuple, will be termed an *orthogonal nonholonomic frame* of $V_n$.

With respect to an orthogonal nonholonomic frame of $V_n$, we have

**Theorem (2.1).** We have

(2.5) $$h_{ij}=\delta_{ij},\quad h^{ij}=\delta^{ij};\quad \overset{i}{e}^{\nu}=\underset{i}{e}^{\nu},\quad \overset{j}{e}_{\lambda}=\underset{j}{e}_{\lambda}.$$

**Theorem (2.2).** The tensors $h_{\lambda\mu}$, $h^{\lambda\mu}$, and $\delta_{\lambda}^{\nu}$ may be expressed in terms of $\underset{i}{e}$, as follows:

(2.6) $$h_{\lambda\mu}=\sum_{i}\underset{i}{e}_{\lambda}\underset{i}{e}_{\mu},\quad h^{\lambda\mu}=\sum_{i}\underset{i}{e}^{\lambda}\underset{i}{e}^{\mu},\quad \delta_{\lambda}^{\nu}=\sum_{i}\underset{i}{e}_{\lambda}\underset{i}{e}^{\nu}.$$

## 3. CHARACTERISTIC ORTHOGONAL NONHOLONOMIC FRAMES.

Let $\underset{i}{e}^{\lambda}$ be unit eigenvectors determined by a symmetric covariant tensor $a_{\lambda\mu}$. Then they satisfy

(3.1) $$(a_{\lambda\mu}-\underset{i}{M}h_{\lambda\mu})\underset{i}{e}^{\lambda}=0,\quad (\underset{i}{M}\text{: scalars}).$$

It is assumed that *the characteristic equation of* (3.3) *has n different real roots* $\underset{i}{M}$, so that we have n different mutually orthogonal unit eigenvectors $\underset{i}{e}^{\lambda}$, $(n=1,\cdots,n)$. The nonholonomic frame in $V_n$ constructed by these eigenvectors $\underset{i}{e}^{\lambda}$ will be called *the characteristic orthogonal nonholonomic frame determined by the tensor* $a_{\lambda\mu}$. Our further discussions will be restricted to the characteristic orthogonal nonholonomic frames only.

For our further discussions, we need the tensors ${}^{(p)}a_{\lambda\mu}$, defined as

(3.2) $$\begin{aligned}&{}^{(1)}a_{\lambda\mu}=a_{\lambda\mu}\\ &{}^{(p)}a_{\lambda\mu}={}^{(p-1)}a_{\lambda\alpha}a_{\mu}{}^{\alpha},\end{aligned}\quad p=2,3\cdots\cdots,$$

A simple inspection shows that ${}^{(p)}a_{\lambda\mu}$ is symmetric.

**Lemma (3.1).** Every eigenvector $\underset{i}{e}^{\lambda}$ of $a_{\lambda\mu}$ is also an eigenvector of the tensor ${}^{(p)}a_{\lambda\mu}$, $(p=2,3,\cdots)$.

**Proof.** We prove our assertion by induction on p. First, we have according to (3.1)

$${}^{(p)}a_{\lambda\mu}\underset{i}{e}^{\lambda}=a_{\lambda\alpha}a_{\mu}{}^{\alpha}\underset{i}{e}^{\lambda}=(a_{\lambda\alpha}\underset{i}{e}^{\lambda})a_{\mu}{}^{\alpha}=(\underset{i}{M}h_{\lambda\alpha}\underset{i}{e}^{\lambda})a_{\mu}{}^{\alpha}=\underset{i}{M}(a_{\lambda\mu}\underset{i}{e}^{\lambda})=\underset{i}{M}^{2}h_{\lambda\mu}\underset{i}{e}^{\lambda},$$

which proves our assertion for the case p=2. Assume that it is true for the case p=m−1. We

(*) Throughout the present paper, Greek indices take values 1, 2, ⋯, n unless explicitly stated otherwise and follow the summation convention, while Roman indices are used for the nonholonomic components of a tensor and run from 1 to n. Roman indices also follow the summation convention.

then have, in a similar manner

$$({}^{(m)}a_{\lambda\mu}-M^{m}h_{\lambda\mu})\underset{i}{e}^{\lambda}=0, \tag{3.3}$$

Which again proves our assertion for the case p=m. Therefore, $\underset{i}{e}^{\lambda}$ is an eigenvector of ${}^{(p)}a_{\lambda\mu}$.

**Theorem (3.2).** The nonholonomic components of ${}^{(p)}a_{\lambda\mu}$ are

$${}^{(p)}a_{x}{}^{i}=\underset{x}{M}{}^{p}\delta_{x}^{i},\quad \text{or}\quad {}^{(p)}a_{xi}=\underset{x}{M}{}^{p}\delta_{xi},\quad (p=1,2,3,\cdots). \tag{3.4}$$

**Proof.** Multiplying both sides of (3.3) by $\underset{j}{e}^{\mu}$ and using (2.4)a and (2.5), we have

$${}^{(p)}a_{\lambda\mu}\underset{i}{e}^{\lambda}\underset{j}{e}^{\mu}=\underset{i}{M}{}^{p}h_{\lambda\mu}\underset{i}{e}^{\lambda}\underset{j}{e}^{\mu},\quad \text{or}\quad {}^{(p)}a_{ij}=\underset{i}{M}{}^{p}\delta_{ij},$$

which shows the second relation of (3.4).

**Theorem (3.3).** The tensor ${}^{(p)}a_{\lambda\mu}$ may be expressed in terms of $\underset{i}{e}^{\lambda}$, as follows:

$${}^{(p)}a_{\lambda\mu}=\sum_{i}\underset{i}{M}{}^{p}\underset{i}{e}_{\lambda}\underset{i}{e}_{\mu},\quad (p=1,2,3,\cdots). \tag{3.5}$$

**Proof.** By means of (3.4), (2.4)b, (2.5), our assertion follows as:

$${}^{(p)}a_{\lambda\mu}={}^{(p)}a_{ij}\overset{i}{e}_{\lambda}\overset{j}{e}_{\mu}=\sum_{i,j}\underset{i}{M}{}^{p}\delta_{ij}\overset{i}{e}_{\lambda}\overset{j}{e}_{\mu}=\sum_{i}\underset{i}{M}{}^{p}\overset{i}{e}_{\lambda}\overset{i}{e}_{\mu}=\sum_{i}\underset{i}{M}{}^{p}\underset{i}{e}_{\lambda}\underset{i}{e}_{\mu}.$$

**Remark.** In case of p=1, Theorem (3.3) represents the well known expression of the tensor $a_{\lambda\mu}$ in Riemannian geometry.

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