An Iterative Method for Location Allocation Problem

Kee Young Kim*

This paper is to present an iterative method for determining an optimum geographic location pattern for the warehouses (sources) that are employed to serve or supply, known quantities of its finished product to a set of customer areas (destinations) that have fixed and known locations, where the number of sources is permitted to vary. What must be determined, in this paper, is the number and location of the sources or warehouses that will minimize the combined costs of transferring and storing the total quantity of products demanded in the destinations or customer areas.

The general questions inquired here are such as

"How many warehouses (sources) should be needed?"

"Where should the sources be located?"

"What customers (destinations) should be serviced by each warehouse?"

given (1) The location of each destination.

(2) The demand requirements of products at each destination.

(3) A set of shipping costs for the transportation links by available transportation modes of interest.

THE MODEL

The function of the location allocation model is to minimize total transportation and storage cost. This can be stated mathematically as follows:

$$\text{Minimize TDC}=\text{TSC}|J+\overline{\text{TTC}}|J \quad \text{for } J=1, 2, \ldots, L \qquad (1)$$

and

* Associate Professor of Management, Yonsei University.

$$\mathrm{TT\overline{C}}|J=\underset{L_J}{\mathrm{Min.}}\ (X_k')\overline{C}_{jk}|L_J \qquad (2)$$

$$\mathrm{TSC}|J=\sum_j S_{j^*}\ X_j \qquad (3)$$

$$\sum_j \sum_i X_{ij}=X_j \qquad (4)$$

$$X_j \geq 0$$

where TDC=total (annual) transportation and storage costs,

$\mathrm{TT\overline{C}}|J$=total transportation cost minimized with respect to warehouse location for each value of $J=1,\ldots\ L$.

(X_k')=a $(1\times K)$ vector whose entries, X_k, represent the quantities of product demanded at each of the K demanding areas or customers.

$\overline{C}_{jk}|L_J$=a vector whose entries, C_{jk}, represent minimized transportation cost between each demanding area k and a specified set of locations, L_J, for J warehouses.

C_{jk}=unit cost of shipping products from area k to warehouse j. [These are defined as $C_{jk}=a_{ij}+b_{jk}$, where a_{ij} is the minimum transportation cost from plant i to warehouse j while b_{jk} stands for the transport cost between warehouse j and customer k. Transportation cost in this study are considered to be a function of distance shipped and unit cost-per-load mile. The cost factors, a_{ij} and b_{jk} in the transportation cost matrix are based on the cheapest transportation mode which might be available for each pair of links in the distribution network.]

L_J=one set of J warehouse, given L possible locations.

$\mathrm{TSC}|J$=total (annual) storage costs for a set of J terminals.

S_{j*}=unit storage cost for the size of terminal j, j^* for each value of J.

X_j=quantity of products passing through terminal j, measured in barrels per year.

The assumptions that will be made for the problems considered here in this model are as follows:

(1) long-run warehouse storage costs are independent of warehouse location. The storage cost function at each location is assumed to be linear with respect to total throughput and to have a positive intercept.

(2) unit storage costs are a function of size of warehouse.

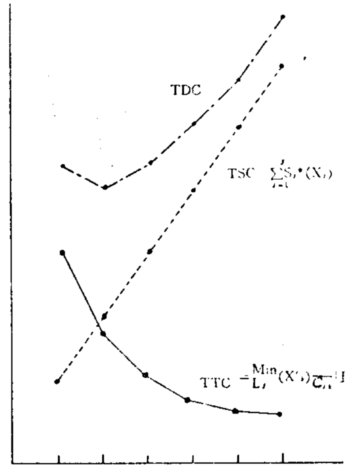
(3) unit transportation costs are independent of total amount of products shipped.

The problem of minimizing Equation (1) with respect to warehouse numbers J and locational pattern L_j can then be solved in two steps. The first step is to obtain the value of $\overline{\mathrm{TTC}}|J$ which is the minimum transportation cost with respect to warehouse locations with varying numbers of sources, J. There are $\binom{L}{J}$ possible combinations of locations L_J out of the population of potential warehouse sites which also are the customer areas. The second step is to compute the total storage costs for a set of J warehouses by multiplying the unit storage cost for the size of each warehouse involved in the set of J warehouses concerned, S_j^* by quantity of products passing through warehouse j, X_j. The sizes of warehouses are to be determined with respect to the optimum locational patterns base on the total demand requirements and their characteristics at each warehouse location.

The sum of $\overline{\mathrm{TTC}}|J$ and $\mathrm{TSC}|J$ with varying numbers of warehouses yields a total cost function minimized with respect to warehouse locations for given numbers of warehouse J. The number of warehouses at the least total costs depends upon the relative slopes of the minimized total transfer cost, $\overline{\mathrm{TTC}}$, and the storage cost, TSC functions as shown in Figure 1.

According to the several recent reviews of quantitative techniques applied to general physical distribution problem based on warehouse location allocation which are presented by Khumawala and Whybark [1] and Bowersox [2]. The techniques that have been used in this field include linear programming [3], mixed integer programming [4], dynamic programming [5], heuristic approaches [6] and simulation models [7].

〈Figure 1〉 **Total Minimum Transportation and Storage Costs with Varying Number of Warehouses.**



The solution procedures presented in the category of analytical methods including L.P., integer programming, guarantee the optimality of the solution whereas the heuristic and simulation methods generate solutions to a problem without the guarantee of an optimal solution even though the Latters offer solutions with less constraints than required for the analitical models.

However, for such a pure location allocation problem as stated in the first step presented above in this study combinatorial optimization procedures have been employed because of their capability of guaranteeing the optimality of the solution [8]. The prblem is to obtain the best assignment of each of the potential customer areas to a warehouses as a source for each possible size of distribution network with different numbers of warehouses within the domain of given number of sources. The best assignment here is referred to as the one resulting in minimum total transportation costs.

The disadvantage of combinatorial procedures lies in the computational burden. The unconstrained problem of selecting r, $(0<r<n)$ facilities from a set of n(number of locations) gives $S(n)$ possible combinations where,

$$S(n)=\sum_{r=1}^{n}\binom{n}{r}=2^n-1.$$

It is evident that solution by direct enumeration is not feasible for realistic sized problems in terms of n. It is the case with other current analytical methods such as integer programming, branch-and-bound methods and linear programming models when the problems are involved in a realistically sized with many staged and multi channeled source-destination patterns.

An iterative method is devised in the present paper to deal with the location allocation problem with guarantee to the same optimality as expected from a combinatorial method at a much less computational burden than that with combinatorial procedures in such a way that there is no limit in feasibility for realistic sized problems.

The iterative algorithm is to be used to determine assignment of customer areas to warehouses by obtaining the minimum total transpotation costs given a set of potential warehouses, $\overline{\mathrm{TTC}}|J$, which was defined in Equation (2) as

$$\overline{\mathrm{TTC}}|J=\underset{L_J}{\mathrm{Min}}(X_k')\overline{C}_{jk}|L_J.$$

Beginning with a transportation cost matrix in a form illustration in Figure 2 the procedure to obtain minimum TTC for J potential terminals is to successively evaluate a shrinking set, based on a descending value for J from $J=L$, the maximum number of potential terminal sites, to $J=1$. In each successive round, the marginal terminal is eliminated whose utilization results in the least cost improvement to the total transportation cost for the given value of J.

In Figure 2, each entry of the transportation cost matrix, C_{jk} stands for unit cost of shipping products from warehouse location j to customer area k. (X_k') is a $(1\times k)$ vector representing the demand requirements of product

⟨Figure 2⟩ Transportation Cost Matrix: C_{jk}

K \ J	Potential Warehouse Locations								(X_K')
	1	2	3	•	•	•	L-1	L	
1	C_{11}	C_{12}	C_{13}	•	•	•	C_{1L-1}	C_{1L}	X_1
2	C_{21}	C_{22}	C_{23}	•	•	•	C_{2L-1}	C_{2L}	X_2
•	•	•	•	•	•	•	•	•	•
K	C_{k1}	C_{k2}	C_{k3}	•	•	•	C_{kL-1}	C_{kL}	X_k

at the customer area k for a given period of time.

The iterative procedure incorporates the following four steps:

Step I. For the first step, when the value of $J=L$, there will be only $\binom{L}{J}=1$ possible combination of location, $L_J|J$. Obtain the $(1\times k)$ vector $\overline{C_{jk}}|L_J$ by scanning C_{jk} by rows and selecting the minimum C_{jk} in each row.

Step II. Calculate $\overline{\mathrm{TTC}}|J$ by multiplying the vector

$$(X_K')=\begin{pmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_k \end{pmatrix} \text{ by the vector } \overline{C}_{jk}|L_J.$$

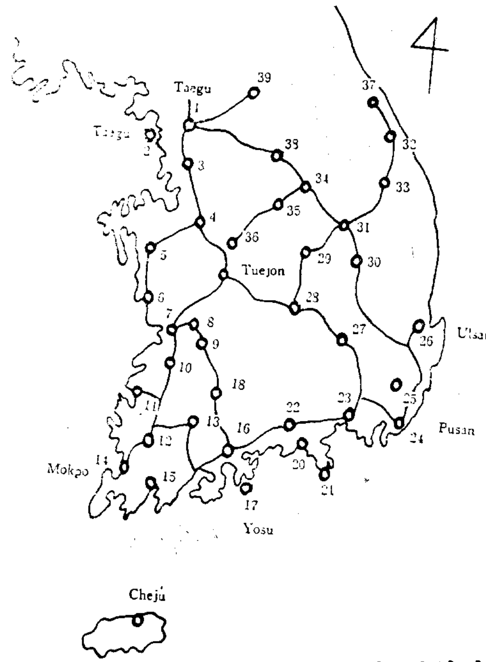
Step III. In general for the values of $J=L-i$ for $i=1, 2\ldots L-1$, there will be $\binom{J+1}{1}$ possible combinations of locations to test the total transport costs. For each L_J there is a sub-matrix $C_{jk}{}^*|L_J$ drawn from the matrix C_{jk}. This sub-matrix will be $(k\times J)$ with the entries of C_{jk} in each of the J columns.

Step IV. Obtain a $(1\times k)$ vector $\overline{C_{jk}}|L_J$ by scanning $C_{jk}{}^*|L_J$ by rows and selecting the minimum C_{jk} in each row of the sub-matrix. Go to Step II to calculate $\overline{\mathrm{TTC}}|J$. For each value of J there are $\binom{J+1}{1}$ values of $\overline{\mathrm{TTC}}|J$, the minimum value of which will identify the marginal terminal site that should be eliminated for the next iteration with new values of J —*i.e.*, the old value decreased by one.

AN EMPIRICAL APPLICATION

As part of the preliminary analysis of an overall production-distribution system design study for a petroleum refining and distributing company in Korea, the actual data based on the company's 1973 operations were collected for an empirical application of the iterative method. The data include (1) total actual demand of products by 40 consuming areas across South Korea which also denoted as potential terminal sites in the form of a (1×40) vector. (2) A transportation cost matrix which specifies the cost of transporting one barrel of refined petroleum product between each possible source and destination in the form of a (40×40) matrix, and (3) A terminal storage cost function in terms of varying size of capacity. A graphical distribution of the 40 consuming areas with identification number is presented in Figure 3.

〈Figure 3〉 Graphical Distribution of the 40 Demand Analysis Areas.



The Optimum Location Allocation With the Minimum Transfer Costs ($\overline{\text{TTC}}|J$)

Using the iterative procedure outlined in the present paper the minimum transportation cost assignments of each of the 40 customer areas to a terminal source for each possible size of distribution network, *i.e.*, for networks with different number of terminals, that is 40, 39, 38, ... 3, 2, 1 nodes are determined.

The computer output of the optimum assignments of customers to a range of 3 to 6 terminal network with $\overline{\text{TTC}}|J$'s for each locational pattern is examplified in Table 1. Each solution assigns each of the 40 individual customer areas to the most economical terminals.

Table 1 Sample Output of Customer Assignments

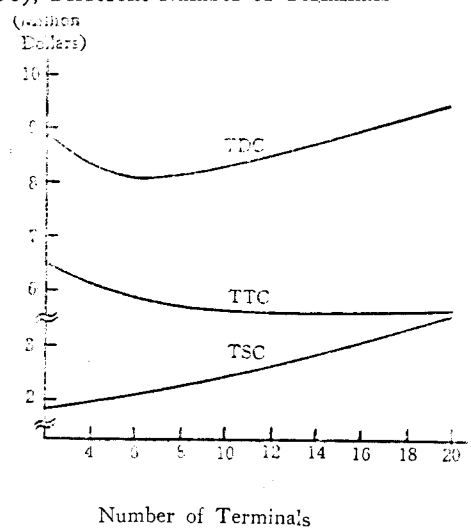
```
TERMINAL CUSTOMERS
   2      1  2  3  4  5 34 35 36 38 39
   7      6  7  8
  17      9 10 11 12 13 14 15 16 17 18 20 22 40
  19     19
  25     21 23 24 25 26 27 28 29 30 31 33
  32     32 37
MIN TTC= 314,564,416.00, COL OUT: 20, MAX TTC=415,514,368.00
# OF COLUMNS: 6
TERMINAL CUSTOMERS
   2      1  2  3  4  5 19 34 35 36 38 39
   7      6  7  8
  17      9 10 11 12 13 14 15 16 17 18 20 22 40
  25     21 23 24 25 26 27 28 29 30 31 33
  32     32 37
MIN TTC= 311,316,480.00, COL OUT: 19, MAX TTC=420,630,784.00
# OF COLUMNS: 5
TERMINAL CUSTOMERS
   2      1  2  3  4  5 19 34 35 36 38 39
  17      6  7  8  9 10 11 12 13 14 15 16 17 18 20 22 40
  25     21 23 24 25 26 27 28 29 30 31 33 32
  32     32 37
MIN TTC=317,757,184.00COL OUT: 7, MAX TTC=423,506,944.00
# OF COLUMNS: 4
TERMINAL CUSTOMERS
   2      1  2  3  4  5 19 34 35 36 37 38 39
  17      6  7  8  9 10 11 12 13 14 15 16 17 18 20 22 40
  25     21 23 24 25 26 27 28 29 30 31 32 33
MIN TTC=330,074,112.00 COL OUT 32 MAX TTC=452,172,544.00
# OF COLUMNS: 3
```

The Total Transportation and Storage Cost (TDC)

Total storage costs of the terminals for varying number of terminals were calculated based on their capacities as determined by the procedure described earlier. Once the total storage costs for the different values of J(the number of terminals) have been calculated, the total transportation and storage costs

(TDC) were computed with respect to the different number of terminals. These total distribution costs (TDC) along with transportation costs($\overline{TTC}|J$)and storage costs (TSC|J) for the best location allocation with varing numbers of terminals are plotted in Figure 4. The total costs decrease gradually from 20 terminals down to 5 terminals, and at 5 terminals turns upward rising at a faster rate than it had declined. These results suggest that the optimum number of terminals is 5 and their locations are 2, 7, 17, 25, 32. The customer areas assigned to each of these five terminals are shown in Table 1.

⟨Figure 4⟩ Transportation Costs (TTC), Total Storage Costs (TSC), and their Sum (TDC), Different Number of Terminals



SUMMARY

An iterative precedure is presented which, given certain assumptions, permits the determination of the optimum location selection of warehouses and the best assignment of customers to each of the selected warehouses with varying numbers of warehouses in such a way as to minimize the total transportations cost. The total distribution costs then can be obtained by combining the total transportation costs and total storage costs computed based on the reflection of economics of scale in facility capacity with respect to different numbers of warehouses as a domain. The total distribution costs, that is, the

total combination of transportation costs and storage costs permit the determination of the optimum number and location warehouse network system in which the economical locational allocation of customers to each of the selected warehouse can be determined.

The iterative method used here to solve for the location pattern of distribution centers appears definitely more efficient in dealing with large size problem with less computational effort without sacrificing the optimality of solution as compared to the existing combinatorial method.

Only a single staged system is considered in the present study, while most real situations rather contain multi-stages in the system in which supply origins such as plants or distribution centers in addition to warehouses are involves. Such a more general situation can be dealt with by an extension of the method presented here.

For a relatively unconstrained production distribution system problem as a comprehensive one the iterative procedure can be used as to obtain a pilot solution which is to define the domain of distribution networks, which is suggestive of the most likely to approximate the optimum system, to be tested with more sophisticated unconstrained approaches such as heuristic and simulation models in such a way that the systems can be evaluated with respect to various criteria other than the single cost measure as used in the analytical models and the optimality of solutions can be improved.

REFERENCES

[1] Khumawala, B. M., and Whybark, D. C., "A Comparison of Some Recent Warehouse Location Techniques," *The Logistics Review*, VII No. 31(1971), pp. 3–19.

[2] Bowersox, Donald J., "Physical Distribution Development, Current Status, and Potential," *Journal of Marketing*, XXXIII (January, 1969), pp. 63–70.

[3] Baumol, W. J. and Wolfe, P., "A Warehouse-Location Problem," *Operations Research,* VI (March-Aprill, 1958),

[4] Efroymson, M. A., and Ray, T. L., "A Branch-Bound Algorithm for Plant Location," *Operations Research,* XIV(May-June, 1966). pp. 361-368.

[5] Chuang, Y. H., and Smith, W. G., "A Dynamic Programming Model for Combined Production, Distribution and Storage," *The Journal of Industrial Engineering,* XV II (January, 1966), pp. 7-13.

[6] Aldrich, Rodney W., "Facility Location: Heuristic Approach," unpublished Ph.D. dissertation, Indiana University, 1973; Kuehn, A. A., and Hamburger, J. M., "A Heuristic Program for Locating Warehouse," *Management Science,* XIX (July, 1963), pp. 643-669.

[7] Gerson, M. L. and Maffei, R. B., "Technical Characteristics of Distribution Simulators," *Management Science,* XII (May, 1966), pp. 670-684; Kim, K. Y., "Planning A Multi-echelon Production-Distribution System in the Petroleum Industry with A Simulation Model," unpublished Ph.D. dissertation, Washington University, 1975.

[8] Stollsteimer, J. F., "A Working Model for Plant Numbers and Locations," *Journal of Farm Economy,* XLV (1963), pp. 631-645.