A Note on Methodologies Used in I-O Forecasting Model

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Introduction

Since the solution vector for input-output forecasting models is not directly obtainable, several iterative procedures have been proposed and utilized. As is often the case in numerical analysis, the question of the consistency between the original system and the converged system of the proposed iteration has been ignored, and no one has tried to express the converged solution explicitly. This paper examines this question and points out the inconsistencies between various well-known iterative procedures used to solve input-output models and the original input-output system.

Let $\{f_t, t=1, \cdots, T\}$ be the given column vectors of size I. Let Q and B (where B is singular) be given matrices of size $I \times I$.[1] Then the I-O forcasting model is, in general, to obtain x_t, $t=1, \cdots, T$ such that

$$x_t = Q[f_t + B(x_{t+1} - x_t)] \tag{1}$$

For $|B|=0$, the set of the solution can not be expressed analytically, and many numerical iterative methods for the solution have been proposed.

This paper analyzes existing iterative procedures and demonstrate that the converged values differ.

(A) Iterative Procedure I

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1) More specifically, f_t is final demand, x_t is output, $Q=(I-A)^{-1}$, where A is input-output coefficient matrix, and B is the capital matrix.

To solve the system of Equation (1), the following iterative procedure is proposed, and we call it Procedure I.

Step (i). Let

$$x_t^{(0)}=Qf_t,\ t=1,\cdots,T \tag{2}$$

Step (ii). Let

$$z_t^{(0)}=B(x_{t+1}^{(0)}-x_t^{(0)}),\quad t=1,\cdots,T-1,\quad \text{and } z_t^{(0)}=\begin{pmatrix} z_{1t}^{(0)} \\ \vdots \\ z_{It}^{(0)} \end{pmatrix} \tag{3}$$

Step (iii). For $i=1,\cdots,I$, regress $z_{it}^{(0)}$ $t=1,\cdots,T-1$ on $(1,\ t,\cdots,t^p)$. That is to find set of maximum likelihood estimators $\{a_{oi}^{(0)},\cdots,a_{pi}^{(0)}\}$ for $\{\alpha_o^{(0)},\cdots,\alpha_{pi}^{(0)}\}$ in the regression model,

$$z_{it}^{(0)}=\sum_{j=0}^{p}\alpha_{ji}^{(0)}t^j+e_{it},\ t=1,\cdots,T-1,\ \ i=1,\cdots,I \tag{4}$$

where error terms $\{e_{it}\}$ are independently and identically distributed according to normal distribution with mean 0 and unknown but common variance. The corresponding representation in multivariate version is;

$$\begin{pmatrix} z_{i1}^{(0)} \\ \vdots \\ \vdots \\ z_{iT-1}^{(0)} \end{pmatrix}=\begin{pmatrix} 1 & 1\cdots\cdots\ 1^p \\ & \vdots \\ 1 & t\cdots\cdots\ t^p \\ & \vdots \\ 1 & T-1\cdots(T-1)^p \end{pmatrix}\begin{pmatrix} \alpha_{oi}^{(0)} \\ \vdots \\ \vdots \\ \alpha_{pi}^{(0)} \end{pmatrix}\begin{pmatrix} e_{i1} \\ \vdots \\ \vdots \\ e_{iT-1} \end{pmatrix} \qquad i=1,\cdots,I$$

where $(e_{i1},\cdots,e_{iT-1})$ is distributed as multivariate normal with zero mean vector and covariance matrix $\sigma^2 I$ with σ^2 unknown and I is the identity matrix of size $(T-1)$.

Step (iv). Define

$$\hat{z}_{it}^{(0)}=\sum_{j=0}^{p}a_{ji}^{(0)}t^j,\quad t=1,\cdots,T,\ i=1,\cdots,I \tag{6}$$

Since $\{a_{ji}^{(0)}\}$ are maximum likelihood estimators for $\{\alpha_{ji}^{(0)}\}$, they can be expressed as:

$$\begin{pmatrix} a_{oi}^{(0)} \\ \vdots \\ a_{pi}^{(0)} \end{pmatrix} = (\Phi'\Phi)^{-1}\Phi' \begin{pmatrix} z_{i1}^{(0)} \\ \vdots \\ z_{iT-1}^{(0)} \end{pmatrix} \qquad i=1,\cdots,I \tag{7}$$

$$\text{where } \Phi = \begin{pmatrix} 1 & 1 & \cdots & 1^p \\ & & & \vdots \\ 1 & t & \cdots & t^p \\ & & & \vdots \\ 1 & T-1 & \cdots & (T-1)^p \end{pmatrix}$$

Step (V). The second iteration proceeds as follows:

Let

$$x_t^{(1)} = Q(f_t + \hat{z}_t^{(0)}),\ t=1,\cdots,T,\ \text{where } z_t^{(0)} = \begin{pmatrix} z_{1t}^{(0)} \\ \vdots \\ z_{It}^{(0)} \end{pmatrix} \tag{8}$$

And return to step (ii), replacing superscript O with 1. The iteration terminates at the k^{th} stage if

$$\max_{\substack{1\le i\le I \\ 1\le t\le T}} \left| \frac{x_{it}^{(k)} - x_{it}^{(k-1)}}{x_{it}^{(k-1)}} \right| < \epsilon$$

where ϵ is a preassinged small number. To express this iterative procedure in matrix notation we define,[1]

$$X_{I\times T} = [x_1 \cdots x_T] = \begin{pmatrix} x_{11} & x_{1T} \\ \vdots \cdots & \vdots \\ x_{I1} & x_{IT} \end{pmatrix}$$

$$X_{I\times(T+1)} = [x_1 \cdots x_T\ x_{T+1}]$$

$$F_{I\times T} = [f_1 \cdots f_T] = \begin{pmatrix} f_{11} \cdots f_{1T} \\ \vdots \quad \vdots \\ f_{I1} \cdots f_{IT} \end{pmatrix}$$

$$C_{T\times(T-1)} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 \cdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & 0 \\ \vdots & \vdots & -1 \\ 0 & 0 & 1 \end{pmatrix},\ \tilde{C}_{(T+1)\times T} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 \cdots\cdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & 0 \\ \vdots & \vdots & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Phi_{(T-1)\times(p+1)} = \begin{pmatrix} 1 & 1 \cdots\cdots & 1^p \\ \vdots & \vdots & \vdots \\ 1 & t \cdots\cdots & t^p \\ \vdots & \vdots & \vdots \\ 1 & (T-1) \cdots & (T-1)^p \end{pmatrix}$$

1) $A_{m\times n}$ denotes a matrix with m rows and n columns.

$$\tilde{\Phi}_{T\times(p+1)}=\begin{bmatrix} & \tilde{\Phi} & \\ 1 & T\cdots T^{p}, \end{bmatrix},\quad \Phi_{(T+1)\times(p+1)}=\begin{bmatrix} & \tilde{\Phi} & \\ 1 & (T+1)\cdots(T+1)^{p} \end{bmatrix}.$$

It is easy to see that (1) can be written as

$$X=Q[F+B\tilde{X}C] \tag{9}$$

Equation (2) through (4) can be expressed as

$$X^{(0)}=QF \tag{10}$$

$$Z^{(0)}=BX^{(0)}C \tag{11}$$

$$Z^{(0)\prime}=\Phi\alpha^{(0)}+\epsilon \tag{12}$$

The estimator $a^{(0)}$ for $\alpha_{(p+1)\times I}{}^{(0)}$ in the multivariate regression model (12) is;

$$a^{(0)}=(\Phi'\Phi)^{-1}\Phi'Z^{(0)\prime}=(\Phi'\Phi)^{-1}\Phi'C'X^{(0)\prime}B' \tag{13}$$

Equation (6) is thus equivalent to

$$\hat{Z}^{(0)\prime}=\tilde{\Phi}a^{(0)} \text{ or } \hat{Z}^{(0)}=BX^{(0)}C\Phi(\Phi'\Phi)^{-1}\tilde{\Phi}'$$

So that Equation (8)

$$X^{(1)}=Q[F+\hat{Z}^{(0)}] \tag{14}$$

or

$$X^{(1)}=Q[F+BX^{(0)}CP] \tag{15}$$

where

$$P=\Phi(\Phi'\Phi)^{-1}\tilde{\Phi}'$$

In general, the relationship between the n^{th} and $(n-1)^{th}$ solution can be expressed as

$$X^{(n)}=Q[F+BX^{(n-1)}CP] \tag{16}$$

Solving (16), where $X^{(0)}=QF$,

we obtain

$$X^{(n)}=\sum_{k=0}^{n}(QB)^{k}(QF)(CP)^{k} \tag{17}$$

Problem of convergence arises prior to that of consistency. For $Q=(I-A)^{-1}$, where A is input output coefficient matrix, the values of components of Q are very small except that diagonal elements are little greater than 1. Values of components of matrix B, the capital coefficient matrix, are very small including many zeroes. Individual country has her own Q and B. The above statement, however, is correct in general. These make not only the elements of matrix QB small but also $(QB)^k$ tending very fast to zero matrix as k becomes large. On the other hand matrix c and $\mathbf{p}$ are uniquely determined, and the speed of convergence of

$$\lim_{k\to\infty}(CP)^{k}$$

is demonstrated in particular cases in Appendix A.

To obtain n^{th} iterative solution of Procedure I, Equation (17) simplifies our work. Specially when we prepare a computer simulation model for I-O forcasting, Equation (17) would be very useful.

Taking the limit of both sides of Equation (16), obtain

$$X=Q[F+BXCP] \tag{18}$$

where

$$X=\lim_{n\to\infty}X^{(n)}$$

Comparing Equation (18) with the original Equation (9), we realize that the proposed iterative procedure actually solves a system of Equation (18) which is different from that for which a solution was sought.[1]

(B) Iterative Procedure II

Another iterative procedure, using derivatives to approximate differences,

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1) See Appendix B for the differences.

is as follow:

Step (i). Let

$$x_t^{(0)}=Qf_t \tag{19}$$

and

$$v_t^{(0)}=Bx_t^{(0)}\ t=1,\cdots,T. \tag{20}$$

Step(ii). Consider the following regression model for $i=1,\cdots,I$,

$$v_{it}^{(0)}=\sum_{k=0}^{p}\Theta_{ki}^{(0)}t^k+e_{it},\ t=1,\cdots,T, \tag{21}$$

and obtain sets of maximum likelihood estimators $\{\Theta_{ki}^{(0)}, k=0,\cdots,p, i=1,\cdots,I\}$ for parameters $\{\Theta_{ki}^{(0)}\}$

Step (iii).

$B(x_{t+1}-x_t)$ is then initially approximated by $w_t^{(0)}$, where

$$w_t^{(0)}=\begin{pmatrix} \frac{d}{dt}\sum_{k=0}^{p}\Theta_{ki}^{(0)}t^k \\ \vdots \\ \frac{d}{dt}\sum_{k=0}^{p}\Theta_{kI}^{(0)}t^k \end{pmatrix}=\begin{pmatrix} \sum_{k=0}^{p}k\Theta_{k1}^{(0)}t^{k-1} \\ \vdots \\ \sum_{k=0}^{p}k\Theta_{kI}t^{k-1} \end{pmatrix} \tag{22}$$

Step (iv). Move to the next iteration, i.e.

Let

$$x_t^{(1)}=Q[f_t+w_t^{(0)}],\ v_t^{(1)}=Bx_t^{(1)},\ \text{and so on} \tag{23}$$

To express Procedure II in matrix notaion, define

$$\overset{D}{\Phi}_{(p+1)\times T}=\begin{pmatrix} 0\cdots\cdots & 0\cdots\cdots & 0 \\ 1 & 1 & 1 \\ 2\cdot 1\cdots & 2t\cdots & 2T \\ \vdots & \vdots & \vdots \\ p1^{p-1} & pt^{p-1} & pT^{p-1}, \end{pmatrix},\quad W=[w_1\cdots\cdots\cdots w_T]$$

Therefore, Equation (19) becomes

$$X^{(0)}=QF, \tag{24}$$

and Equation (20) becomes

$$V^{(0)}=BX^{(0)}$$

Regression Equation (21) is thus equivalent to

$$V^{(0)\prime}\Phi\Theta^{(0)}+\epsilon,$$

from which we obtain

$$\hat{\Theta}^{(0)}=(\tilde{\Phi}'\tilde{\Phi})^{-1}\tilde{\Phi}'V^{(0)\prime} \tag{25}$$

Equation (22) becomes

$$W^{(0)}=\Theta^{(0)\prime}\Phi^{D},$$

from which Equation (23) can be written

$$X^{(1)}=Q[F+BX^{(0)}\Phi(\Phi'\Phi)^{-1}\Phi^{D}]$$

In gentral we obtain

$$X^{(n)}=Q[F+BX^{(n-1)}\Phi(\Phi'\Phi)^{-1}\Phi^{D}] \tag{26}$$

Since

$$X^{(0)}=QF,$$

we obtain

$$X^{(n)}=\sum_{k=0}^{n}(QB)^{k}QFP^{k},$$

where

$$P=\tilde{\Phi}(\tilde{\Phi}'\tilde{\Phi})\Phi^{D}$$

Taking the limit of both sides of (26), we obtain

$$X=Q[F+BXP] \tag{27}$$

Thus the set of equations for which the converged value of (26) is the solution is (27), not (9), as commonly thought. The set of equations solved by Procedure II also differs from that solved by Procedure I, as indicated above.

(C) Iterative Procedure III

Another procedure, which is similar to procedure I, is as follows:

Step (i). Let

$$\hat{\boldsymbol{x}}_t^{(0)}=\boldsymbol{Q}\boldsymbol{f}_t \quad t=1,\cdots,T.$$

Step (ii). Regress $x_{it}^{(0)}$ on $(1,\cdots,t^p)$ and then obtain estimated values $x_{it}^{(0)}$ $i=1,\cdots,I,$ $t=1,\cdots,T+1.$

Step (iii). Let

$\boldsymbol{x}_t^{(1)}=\boldsymbol{Q}[\boldsymbol{f}_t+\boldsymbol{B}(\boldsymbol{x}_{t+1}^{(0)}-\boldsymbol{x}_t^{(0)}]$, and repeat

In matrix notation we obtain:

$$\boldsymbol{X}^{(0)}=\boldsymbol{Q}\boldsymbol{F}$$

$$\boldsymbol{X}^{(0)\prime}=\tilde{\boldsymbol{\Phi}}\boldsymbol{\Theta}^{(0)}+\epsilon$$

$$\tilde{\boldsymbol{X}}^{(0)\prime}=\tilde{\tilde{\boldsymbol{\Phi}}}(\tilde{\boldsymbol{\Phi}}'\tilde{\boldsymbol{\Phi}})^{-1}\tilde{\boldsymbol{\Phi}}'\boldsymbol{X}^{(0)\prime}$$

$$\tilde{\boldsymbol{X}}^{(0)}=\boldsymbol{X}^{(0)}\tilde{\boldsymbol{\Phi}}(\tilde{\boldsymbol{\Phi}}'\tilde{\boldsymbol{\Phi}})^{-1}\tilde{\tilde{\boldsymbol{\Phi}}}'$$

or

$$\boldsymbol{X}^{(1)}=\boldsymbol{Q}[\boldsymbol{F}+\tilde{\boldsymbol{X}}^{(0)}\tilde{\boldsymbol{C}}]$$

$$\boldsymbol{X}^{(1)}=\boldsymbol{Q}[\boldsymbol{F}+\boldsymbol{B}\boldsymbol{X}^{(0)}\boldsymbol{R}\tilde{\boldsymbol{C}}],$$

where

$$R=\tilde{\boldsymbol{\Phi}}(\tilde{\boldsymbol{\Phi}}'\tilde{\boldsymbol{\Phi}})^{-1}\tilde{\tilde{\boldsymbol{\Phi}}}'$$

In general, we obtain

$$X^{(n)}Q=[F+BX^{(n-1)}V] \tag{28}$$

where

$$V=RC.$$

Substituting ${}^{(0)}X=QF$, and solving (28), we obtain

$$X^{(n)}=\sum_{k=0}^{n}(QB)^{k}QFV^{k} \tag{29}$$

As previous cases, taking the limit of both sides of equation (28), we obtain

$$X=Q[F+BXV] \tag{30}$$

It is easy to see that (30), the set of equatins for which the converged value of (29) is the solution, is different not only from the original set of equations for which solution was sought, (9), but also from the set of equations solved by the two alternative procedure discussed above.

Appendix A

For given p and t, $p\leqslant T-1$, we concerned with investigating the behavior of

$$S=C\Phi(\Phi'\Phi)^{-1}\tilde{\Phi}'$$

where matrices C, Φ and $\tilde{\Phi}$ are defined on page 6.
In this section, however, one of simulations is presented. When $p=3$ and $T=10$, as computer output shows,

$$\max_{i,j}|s_{ij}^{(8)}|<\frac{1}{10^{8}}$$

$$\max_{i,j}|s_{ij}^{(16)}|<\frac{1}{10^{17}}$$

$$\max_{i,j}|s_{ij}^{(32)}|<\frac{1}{10^{35}}$$

where $S^{k}=(s_{ij}^{(k)})$, i.e $s_{ij}^{(k)}$ denotes (i,j) component of the matrix S^{k}, which

S

-.85859+00	-.28223+00	.20203-01	.12121+00	.90910-01	-.38743-06	-.80809-01	-.80809-01	.70708-01	.44445+00
.57576+00	-.45455-01	-.30303+00	-.30303+00	-.15152+00	.45455-01	.18182+00	.15152+00	-.15152+00	-.83334+00
.30303+00	.45454-01	-.88745-01	-.12987+00	-.10823+00	-.54112-01	.21650-02	.30303-01	-.32842-06	-.11905+00
.10101+00	.10101+00	.59885-01	-.21643-02	-.64935-01	-.10823+00	-.11183+00	-.55557-01	.80806-01	.31746+00
-.30304-01	.12121+00	.14286+00	.80087-01	-.21645-01	-.11688+00	-.16017+00	-.10606+00	.90908-01	.47619+00
-.90911-01	.10606+00	.16017+00	.11688+00	.21646-01	-.80086-01	-.14286+00	-.12121+00	.30307-01	.35307-01
-.80808-01	.55556-01	.11183+00	.10823+00	.94935-01	.21644-02	-.59885-01	-.10101+00	-10101+00	-.39682-01
.52154-06	-.30303-01	-.21656-02	.54111-01	.10822+00	.12987+00	.88742-01	-.45458-01	-.30304+00	-.71429+00
.15152+00	-.15152+00	-.18182+00	-.45456-01	.15151+00	.30303+00	.30303+00	.45454-01	-.57576+00	-.16667+01
-.70789-01	.80808-01	.80889-81	.14805-85	.90907-01	-.12121+00	-.20198-01	.28283+00	.85860+00	.17777+01

S²

.57576+00	.30303+00	.10101+00	-.30304-01	-.90909-01	-.80807-01	.27509-05	.15152+00	.37374+00	.66668+00
-.62122+00	-.25758+00	.62212-06	.15152+00	.19697+00	.13636+00	-.30308-01	-.30104+00	-.68183+00	-.11667+01
-.25758+00	-.13420+00	-.41125-01	.21645-01	.54113-01	.56277-01	.28138-01	-.30304-01	-.11905+05	-.23810+00
.67824-06	-.41126-01	-.62049-01	-.62771-01	-.43290-01	-.36070-02	.56278-01	.13637+00	.23666+00	.35715+00
.15152+00	.21645-01	-.62771-01	-.10173+00	-.95238-01	-.43289-01	.54115-01	.19697+00	.38529+00	.61906+00
.19697+00	.54113-01	-.43290-01	-.95238-01	-.10173+00	-.62769-01	.21648-01	.15152+00	.32685+00	.54763+00
.13636+00	.56277-01	-.36077-02	-.43291-01	-.62771-01	-.62050-01	-.41126-01	.22317-06	.61328-01	.14286+01
-.30304-01	.28138-01	.56277-01	.54112-01	.21644-01	-.41128-01	-.13420+00	-.25758+00	-.31126+00	-.59525+00
-.30303+00	-.30303-01	.13636+00	.19697+00	.15151+00	-.46529-05	-.25758+00	-.62123+00	-.10909+01	-.16667+01
.15152+00	.36508-06	-.80808-01	-.90908-01	-.30300-01	.10102+00	.30304+00	.57577+00	.91921+00	.13334+01

S⁴

.70108-01	.70708-01	.70708-01	.70708-01	.70709-01	.70709-01	.70710-01	.70710-01	.70710-01	.70711-01
-.10606+00	-.10606+00	-.10606+00	-.10606+00	-.10606+00	-.10606+00	-.10607+00	-.10607+00	-.30303-01	- 30303-01
-.30303-01	-.30304-01	-.30304-01	-.30304-01	-.30304-01	-.30304-01	-.30305-01	.20202-01	.20202-01	.20202-01
.20203-01	.20203-01	.20203-01	.20203-01	.20203-01	.20203-01	.20203-01	.45455-01	.45455-01	.45455-01
.45456-01	.45456-01	.45456-01	.45456-01	.45457-01	.45457-01	.45457-01	.45455-01	.45455-01	.45455-01
.45456-01	.45456-01	.45456-01	.45456-01	.45457-01	.45457-01	.45457-01	.20202-01	.20202-01	.20202-01
.20202-01	.20203-01	.20203-01	.20203-01	20203-01	.20203-01	.20203-01	-.30303-01	-.30304-01	-.30304-01

−.30303−01	−.30304−01	−30304−01	−.30304−01	−.30304−01	−.30304−01	−.30304−01	−.30305−01	−.30305−01	−.30305−01
−.10606+00	−.10606+00	−.10606+00	−.10606+00	−.10606+00	−.10606+00	−.10606+00	−.10607+00	−.10607+00	−.10607+00
.70708−01	.70708−01	.70709−01	.70709−01	.70709−01	.70710−01	.70710−01	.70710−01	.70711−01	.70711−01
	S^8								
.18044−08	.16298−08	.14552−08	.16298−08	.14552−08	.11642−08	.13388−08	.29104−09	−.58208−10	−.34925−09
−.26776−08	−.23865−08	−.19791−08	−.22701−08	−.20373−08	−.16880−08	−.18044−08	−.23283−09	.23283−09	.69849−09
−.72760−09	−.69849−09	−.55297−09	−.64028−09	−.55297−09	−.46566−09	−.52387−09	.00000	.87311−10	.20373−09
.48021−09	.40745−09	.36830−09	.39290−09	.39290−09	.30559−09	.30559−09	.29104−10	−.29104−10	−.20373−09
.40745−09	.34925−09	.17462−09	.26193−09	.11642−09	−.58208−10	.87311−10	−.94028−09	−.84401−09	−.96043−09
.16880−08	.16298−08	.15134−08	.15425−08	.14552−08	.13388−08	.13097−08	.64028−09	.49477−09	.32014−09
.65484−09	.62573−09	.55297−09	.61118−09	.58028−09	.52387−09	.52387−09	.21828−09	.11642−09	.10186−09
−.46566−09	−.40745−09	−.26193−09	−.40745−09	−.29104−09	−.17462−09	−.20373−09	.26193−09	.34925−09	.49477−09
−.33760−08	−.30850−08	−.26193−08	−.28522−08	−.26193−08	−.23865−08	−.23865−08	−.81491−09	−.46566−09	.00000
.28522−08	.27358−08	.25029−08	.27940−08	.25029−08	.22701−08	.23283−08	.12806−08	.98953−09	.75670−09
	S^{16}								
.11155−17	.10478−17	.11469−17	.10164−17	.97748−18	.64036−18	.83009−18	−.24564−19	−.35914−18	−.53617−18
−.16382−17	−.15153−17	−.16136−17	−.14289−17	−.13697−17	−.87668−18	−.11511−17	.18381−18	.66916−18	.97234−18
−.41589−18	−.38540−18	−.41335−18	−.36084−18	−.34728−18	−.21769−18	−.28630−18	.63527−19	.18635−18	.26682−18
.79198−19	.63316−19	.92962−19	.42140−19	.48069−19	−.29858−19	.20964−19	−.16242−18	−.23548−18	−.27719−18
.21345−18	.52940−19	−.14442−18	−.15882−18	−.13934−18	−.20964−18	−.20795−18	−.75767−18	−.83263−18	−.10812−17
.97663−18	.10313−17	.12972−17	.11575−17	.11041−17	.80849−18	.99823−18	.54252−18	.26512−18	.26639−18
.65222−18	.65222−18	.70812−18	.67720−18	.63400−18	.49255−18	.57090−18	.23251−18	.89795−19	.43622−19
−.11393−18	−.47434−19	−.16941−19	−.22023−19	−.22023−19	.63527−19	.28799−19	.39514−18	.47519−18	.61452−18
−.20024−17	−.20015−17	−.23522−17	−.21083−17	−.20100−17	−.14137−17	−.17796−17	−.55311−18	.16941−19	.17703−18
.19399−17	.20363−17	.24954−17	.22836−17	.21701−17	.16365−17	.19753−17	.11316−17	.62511−18	.62177−18
	S^{32}								
.19534−36	.86814−37	−.67097−38	−.53484−37	−.13384−36	−.97251−37	−.54505−36	−.64764−36	−.81623−36	−.18002−36
.53589−38	.19181−36	.25415−36	.22197−36	.34312−36	.30592−36	.98963−36	.11218−35	.14028−35	−.32028−37
.18191−37	.71706−37	.87669−37	.78525−37	.10533−36	.97894−37	.27415−36	.30529−36	.38057−36	−.90944−37

−.90144−37	−.13532−36	−.20678−36	−.20354−36	−.19059−36	−.17996−36	−.1474 −36	−.28660−36	−.27918−36	−.34148−36
−.45769−36	−.61899−36	−.89837−36	−.87548−36	−.82134−36	−.75670−36	−.82055−36	−.10978−35	−.10390−35	−.12569−35
.51672−36	.52718−36	.63902−36	.57641−36	.55225−36	.41383−36	.49382−36	.26334−36	.12198−36	.11637−36
.27669−36	.26728−36	.31045−36	.27200−36	.26275−36	.18076−36	.22550−36	.49364−37	−.34063−37	−.60262−37
.23458−36	.33030−36	.48576−36	.47711−36	.44702−36	.41703−36	.45019−36	.62988−36	.60524−36	.73332−36
−.70868−36	−.62740−36	−.66617−36	−.55962−36	−.54712−36	−.32006−36	−.43452−36	.18887−36	.42476−36	.57466−36
.10699−35	.11063−35	.13553−35	.12276−35	.11748−35	.89222−36	.10578−35	.62210−36	.33451−36	.34290−36

S^{64}

.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

is k^{th} power of S. We realize through simulations the larger p and S becomes the faster S^k converges.

Appendix B

We are concerned with searching differences between the system

$$X = Q[F + B\tilde{X}\tilde{C}]$$

and the system

$$X = Q[F + BXC\Phi(\Phi'\Phi)^{-1}\tilde{\Phi}'].$$

To simplify the problem we investigate differences between

$$\tilde{X}\tilde{C}$$

and

$$XC\Phi(\Phi'\Phi)^{-1}\tilde{\Phi}'$$

in several specified cases.

Case (i). $I=3,\ p=2,\ T=4$

Through straightforward computations, it is easy to obtain

$$C\Phi(\Phi'\Phi)^{-1}\tilde{\Phi}' = \begin{pmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 4 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 1 & 3 \end{pmatrix},$$

so that we obtain

$$XC\Phi\ (\Phi'\Phi)^{-1}\tilde{\Phi}' = \begin{pmatrix} X_{12}-X_{11}, & X_{13}-X_{12}, & X_{14}-X_{13}, & 4X_{12}-X_{11}+3X_{14}-6X_{13} \\ X_{22}-X_{21}, & X_{23}-X_{22}, & X_{24}-X_{23}, & 4X_{22}-X_{21}+3X_{24}-6X_{23} \\ X_{32}-X_{31}, & X_{33}-X_{32}, & X_{34}-X_{33}, & 4X_{32}-X_{31}+3X_{34}-6X_{33} \end{pmatrix}$$

which is comparable to.

$$\tilde{X}\tilde{C} = \begin{pmatrix} X_{12}-X_{11}, & X_{13}-X_{12}, & X_{14}-X_{13}, & X_{15}-X_{14} \\ X_{22}-X_{21}, & X_{23}-X_{22}, & X_{24}-X_{23} & X_{25}-X_{24} \\ X_{32}-X_{31}, & X_{33}-X_{32}, & X_{34}-X_{33} & X_{35}-X_{34} \end{pmatrix}$$

Note

First 3 columns are identical each other. But in the last column X_{i5} is substituted with $4X_{i2}-X_{i1}+2X_{i4}-6X_{i3}$ for $i=1, 2, 3$.

Case (ii). $I=3,\ p=2,\ T=5$

We obtain

$$C\Phi(\Phi'\Phi)^{-1\prime}\tilde{\Phi}=\frac{1}{80}\begin{pmatrix} -76 & -12 & 12 & -4 & -60 \\ 64 & -32 & -48 & 16 & 160 \\ 24 & 8 & -8 & -24 & -40 \\ -16 & 48 & 32 & -64 & -240 \\ 4 & -12 & 12 & 76 & 180 \end{pmatrix}$$

so that the 1st column of 1st row of $XC\Phi(\Phi'\Phi)^{-1}\tilde{\Phi}'$ becomes

$$\frac{1}{80}(-76X_{11}+64X_{12}+24X_{13}-16X_{14}+4X_{15})$$

which is comparable to

$$X_{12}-X_{11}$$

which is the 1st column of 1st row XC. The differences in the remainders are obviously clear.

REFERENCE

[1] Almon, Clopper, Jr., *The American Econony to 1975: An Inter-industry Forecast*, New York: Harper and Row, undated.