# OPTIMAL STOPPING IN SAMPLING FROM A MULTIVARIATE DISTRIBUTION

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**Abstract**

Optimal stopping problem without recall from a multivariate distribution is solved by using the concept of an equilibrium point which was introduced by J. Nash. The solution is derived for the two cases:

1. The case where the observation cost C is positive and the given upper bound K on the number of observations is infinite.
2. The case where the observation cost C is zero and the given upper bound K on the number of observations is finite.

## 1. Introduction

The optimal stopping problem without recall in which the experimenter observes the sequential random sample from a specified bivariate distribution has been studied by the Sakaguchi [2]. This note is an extension of the Sakagn hi's results for the multivariate random variables.

Let $(X_{1i},\ldots,X_{ni})$ $i=1,2,\ldots$ be independently and identically distributed multivariate random variables that can be observed sequentially at a cost of $c_1,\ldots,c_n$ $(c_i\geq 0)$ per observation of $X_1,\ldots,X_n$, respectively. We assume that the experimenter knows the joint distribution function H $(x_1,\ldots,x_n)$, and that if the experimenter terminate the sampling process after having observed the values $(X_{1i}=x_{1i},\ldots,x_{ni}=x_{ni})$ for $i=1,\ldots,m$, then his gain is a set of values $X_{1m}-mc_1,\ldots,X_{nm}-mc_n$. We shall suppose that there is a given upper bound $K$ $(2\leq K\leq\infty)$ on the number of observations that can be taken. We shall introduce the concept of equilibrium neutral values and derive the explicit solution by using it, for the case where $c_i>0$ and $K=\infty$ in section 2, and $c_i=0$ and $K<\infty$ in section 3. For terminology and notation, we follow [2]

## 2. Optimal stopping problem for the case where $c_i>0$ and $K=\infty$

Let $(u_1,\ldots,u_n)$ and $\tau=\tau(u_1,\ldots,u_n)$ denote respectively a pair of neutral values such that the

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sampling procedure is terminated at the first m such that $X_{1m} \geq u_1, \ldots, X_{nm} \geq u_n$ and the random stopping time when the neutal values $(u_1, \ldots, u_n)$ are used. Then the expected net gain from the observations of $X_i$, when using the neutral values $(u_1, \ldots, u_n)$, is defined by

(2.1) $M_i \ (u_1, \ldots, u_n) = E[X_{i\tau} - \tau c_i | u_1, \ldots, u_n] \quad i = 1, 2, \ldots, n.$

An equilibrium point $(u_1^*, \ldots, u_n^*)$ for the functions $M_1, \ldots, M_n$ satisfies the following:

(2.2) $M_i(u_1^*, \ldots, u_n^*) = \max_{u_i} M_i(u_1^*, \ldots, u_i, \ldots, u_n^*) \quad i = 1, 2, \ldots, n.$

The problem is to find the equilibrium point $(u_1^*, \ldots, u_n^*)$ in which each value $u_i^*$ maximizes the function $M_i \ (i = 1, \ldots, n)$. J. Nash showed the existence and solvability of such equilibrium point [1].

Theorem 1: Let S denote the event $\{(x_1, \ldots, x_n) | x_1 \geq u_1, \ldots, x_n \geq u_n\}$. Then we have following:

(2.3) $M_i \, (u_1, \ldots, u_n) = E[X_i | S] - \dfrac{c_i}{Pr\{S\}}, \qquad i = 1, \ldots, n.$

The equilibrium point $(u_1^*, \ldots, u_n^*)$ is the solution of the following simultaneous equations:

(2.4) $E[X_i - u_i | S] = \dfrac{c_i}{Pr\{S\}}, \qquad i = 1, 2, \ldots, n.$

and we have

(2.5) $M_i(u_1^*, \ldots, u_n^*) = u_1^* \qquad i = 1, 2, \ldots, n.$

Proof: $M_i(u_1 \ldots, u_n) = E[X_{i\tau} - \tau c_i]$

$$= \sum_{m=1}^{\infty} (1 - Pr\{S\})^{m-1} \ Pr\{S\} E[X_i - m c_i | S]$$

$$= Pr\{S\} \Big[ E[X_i | S] \sum_{m=1}^{\infty} (1 - Pr\{S\})^{m-1} - c_i \sum_{m=1}^{\infty} m (1 - Pr\{S\})^{m-1} \Big]$$

$$= Pr\{S\} \Big[ \frac{E[X_i | S]}{Pr\{S\}} - \frac{c_i}{(Pr\{S\})^2} \Big]$$

$$= E[X_i | S] - \frac{c_i}{Pr\{S\}} \qquad i = 1, 2, , \ldots, n.$$

Therefore, the eq. (2.3) is proved.

Let the probability density function of $H(x_1, \ldots, x_n)$ be $h(x_1, \ldots, x_n)$, then the eq. (2.3) is rewritten by

$$M_i(u_1, \ldots, u_n) = \frac{\int_{u_1}^{\infty} \cdots \int_{u_n}^{\infty} x_i h(x_1, \ldots, x_n) \ dx_n \ldots dx_1 - c_i}{\int_{u_1}^{\infty} \cdots \int_{u_n}^{\infty} h \, (x_1, \ldots, x_n) \ dx_n \ldots dx_1},$$

Since $M_i \ (u_1, \ldots, u_n)$ has its maximum value when $\dfrac{\partial M_i}{\partial u_i} = 0$, we can easily obtain

(2.6) $\int_{u_1}^{\infty} \cdots \int_{u_n}^{\infty} (x_i - u_i) \ h \ (x_1, \ldots, x_n) \ dx_n \ldots dx_1 = c_i \qquad i = 1, 2, \ldots, n.$

Equation (2.6) is equivalent to eq. (2.4), thus we know that eq. (2.4) is hold.

Let $S^* = \{(x_1, \ldots, x_n) | x_1 \geq u_1^*, \ldots, x_n \geq u_n^*\}$,

then we have

$$M_i(u_1^*, \ldots, u_n^*) = E[X_i | S^*] - \frac{c_i}{Pr\{S^*\}}$$

$$= E[X_i | S^*] - E[X_i - u_i^* | S^*]$$

$$= u_i^*, \qquad i = 1, 2, \ldots, n.$$

Hence the theorem is proved completely.

## 3. Optimal stopping problem for the case where $c_i=0$ and $K<\infty$

Let $(X_{1m},\ldots,X_{nm})$ be the mth observed value. We consider a class of stopping rules in which the experimenter has a set of neutral values $\{u_{1j}\}_{j=1}^{k-1},\ldots,\{u_{nj}\}_{j=1}^{k-1}$ (abbreviated by $u_1^{k-1},\ldots,u_n^{k-1}$, respectively) such that the sampling is terminated at the first mth observation satisfying $X_{1m}\geq u_{1m},\ldots,X_{nm}\geq u_{nm}$.

Let $\tau=\tau(u_1^{k-1},\ldots,u_n^{k-1})$ denote the random stopping time when the neutral values $u_1^{k-1},\ldots,u_n^{k-1}$ are used.

Let

(3.1) $M_k^{(i)}(u_1^{k-1},\ldots,u_n^{k-1})=E[X_{i\tau}\mid u_1^{k-1},\ldots,u_n^{k-1}]$ $\qquad i=1,2,\ldots,n$

be the expected net gain from the observations of $X_i$. We shall determine the equilibrium points $\{(u_{1j}^*,\ldots,u_{nj}^*)\}$ as follows:

First set

(3.2) $u_{i1}^*=\mu_{i1}=E[X_i]$ $\qquad i=1,2,\ldots,n,$

and we determine the sequence of values $\{u_{1j}^*\}_{j=1}^{m-1},\ldots,\{u_{nj}^*\}_{j=1}^{m-1}$ (abbreviated by $u_1^{*m-1},\ldots,u_n^{*m-1}$, respectively). Then the successive equilibrium point $(u_1^{*m},\ldots,u_n^{*m})$ is determened by the following equations:

(3.3) $M_{m+1}^{(i)}(u_1^{*m},\ldots,u_n^{*m})$

$=\max_{u_{im}} M_{m+1}^{(i)}(u_1^{*m};\ldots;u_i^{*m-1},u_{im};\ldots;u_n^{*m})$ $\qquad i=1,2,\ldots,n.$

Let $M_k^{(i)}$ and $M_k^{(i)*}$ $(i=1,2,\ldots,n;k=1,2,\ldots)$ denote $M_k^{(i)}(u_1^{k-1},\ldots,u_n^{k-1})$ for any infinite sequences of numbers $\{u_{1j}\}_{j=1}^{\infty},\ldots,\{u_{nj}\}_{j=1}^{\infty}$ and $M_k^{(i)}(u_1^{*k-1},\ldots,u_n^{*k-1})$ respectively.

Then we have following theorem.

Theorem 2: $M_k^{(i)}$ $(i=1,\ldots,n)$ satisfy

(3.4) $M_{k+1}^{(i)}=M_k^{(i)}+\int_{u_{1k}}^{\infty}\cdots\int_{u_{nk}}^{\infty}(x_i-M_k^{(i)})\,h(x_1,\ldots,x_n)\,dx_n\ldots dx_1,$

and

$M_1^{(i)}=u_{i1}$ $\qquad i=1,2,\ldots,n;K=1,2,\ldots.$

Let $\{\mu_{ij}\}_{j=1}^{\infty}$ $(i=1,2,\ldots,n)$ be the infinite sequences of numbers defined by the following simultaneous recurrence relations.

(3.5) $\mu_{i,j+1}=\mu_{ij}+\int_{\mu_{1j}}^{\infty}\cdots\int_{\mu_{nj}}^{\infty}(x_i-\mu_{ij})\,h(x_1,\ldots,x_n)\,dx_n\ldots dx_1$ $\quad i=1,2,\ldots n;j=1,2,\ldots.$

Then the successive equilibrium points $(u_{1k}^*,\ldots,u_{nk}^*)$ satisfying eq. (3.3) are given by

(3.6) $u_{ik}^*=\mu_{ik}$ $\qquad i=1,2,\ldots,n;K=1,2,\ldots.$

And we have

(3.7) $M_{k+1}^{(1)*}=\mu_{1,k+1},\ldots,M_{k+1}^{(n)*}=\mu_{n,k+1}.$

Proof: From the eq. (3.1), we have

$$M_{k+1}^{(i)}=(1-Pr\{X_{11}\geq u_{1k},\ldots,X_{n1}\geq u_{nk}\})M_k^{(i)}+\int_{u_{1k}}^{\infty}\cdots\int_{u_{nk}}^{\infty}x_ih(x_1,\ldots,x_n)\,dx_n\ldots dx_1$$

$$=M_k^{(i)}+\int_{u_{1k}}^{\infty}\cdots\int_{u_{nk}}^{\infty}(M_k^{(i)}-x_i)\,h(x_1,\ldots,x_n)dx_n\ldots dx_1 \qquad i=1,2,\ldots,n.$$

We shall prove eqs. (3.6) and (3.7) by mathematical induction. From eqs. (3.4) and (3.5), we have

$$M_2^{(i)*}=u_{i1}^*+\int_{u11}^{\infty}\cdots\int_{un1}^{\infty}(x_i-u_i1)\,h(x_1,\ldots,x_n)\,dx_n\ldots dx_1$$

$$=\mu_{i1}+\int_{\mu11}^{\infty}\cdots\int_{\mu n1}^{\infty}(x_i-\mu_{i1})\,h(x_1,\ldots,x_n)\,dx_n\ldots dx_1$$

$$=\mu_{i2}.$$

Equations (3.6) and (3.7) are valid for $j=1$. Assume that they are valid up to $k-1$, then by eq. (3.4) we have following equation;

$$M_{k+1}^{(i)}(u_1^{**k};\ldots;u_i^{**k-1};u_{ik};\ldots;u_n^{**k})$$

$$=M_k^{(i)*}+\int_{u_{1k}^*}^{\infty}\cdots\int_{u_{ik}}^{\infty}\cdots\int_{u_{nk}^*}^{\infty}(x_i-M_k^{(i)*})\,h\,(x_1,\ldots,x_n)\,dx_n\ldots dx_1$$

$$=\mu_{ik}+\int_{u_{1k}^*}^{\infty}\cdots\int_{u_{ik}}\cdots\int_{u_{nk}^*}^{\infty}(x_i-\mu_{ik})\,h\,(x_1,\ldots,x_n)\,dx_n\ldots dx_1.$$

$M_{k+1}^{(i)}$ has its maximum when $\dfrac{\partial M_{k+1}^{(i)}}{\partial u_{ik}}=0$, which gives us

$$u_{ik}^*=\mu_{ik} \text{ and } M_{k+1}^{(i)*}=\mu_{i,k+1}$$

$$i=1,2,\ldots,n\,;\,k=1,2,\ldots.$$

Therefore, eqs. (3.6) and (3.7) are valid for all $k$.
Hence the theorem is proved completely.

## References


1. Nash, J., "Non-cooperative games." Ann. of Math. Vol. 54 (1951), pp.286–295.
2. Sakaguchi, M., "Optimal stopping in sampling from a bivariate distribution." J.O.R.S., Vol. 16, No. 3, pp.186–200.