Technical Paper

An Axisymmetrical Dock in Waves

by

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Abstract

Linearized motions of an axisymmetrical dock freely floating in a regular plane wave are discussed. An extension of the Bessho variational principle (Bessho [3]) is derived to obtain a numerical procedure for a solution of the boundary value problem associated with the fluid motion.

The added mass and the damping coefficients for a circular dock in vertical (heave) and horizontal (surge) oscillations are evaluated numerically, and the results seems to be satisfactory.

1. Introduction

Motions of an axisymmetrical dock freely floating on a free surface of water are important for designs of artificial islands or ocean platforms. The hydrodynamic properties of a floating semi-sphere were discussed by Havelock [6] and Barakat [2] and those of semispheroids by Kim [9] and Sao, Maeda & Hwang [14]. Miles & Gilbert [10], Garrett [5], Miles [11] and Black, Mei & Bray [4] discussed wave forces on a circular dock. In the present work, a general variational formulation for radiation and scattering of water waves by axisymmetrical dock is obtained as an extension of the Bessho variational principle (Bessho [3], Isshiki [7,8], Mizuno [12], Sao, Maeda & Hwang [14].)

Sao et al, also discussed heave oscillations of a circular dock on the basis of the Bessho variational principle and slender body assumptions. But their method seems to be out of their assumptions for this problem. In this paper, the same problem is dealt with from a little different point of view. An extension of the Bessho variational principle is obtained, and the Rayleigh-Ritz procedure is derived based on the variational formulation and an admissible potential similar to that by Miles & Gilbert [10].

The added mass and the damping coefficients are calculated for a circular dock in heave and surge oscillations. The numerical results seem to be reasonable, and are compatible with the results for a sphere and spheroids.

Near the end of the preparation of this paper, the authors received an interesting paper from Dr. K.J. Bai [1] which also discusses linear water wave problems from the standpoint of variational calculus. He regards the Sommerfeld radiation condition as a boundary condition on a circular cylindrical surface which is assumed to be located far from a body, and approximates the boundary value problem in an infinite region by a problem in a finite fluid region within the cylindrical surface.

2. Linearized theory of motion of an axisymmetrical dock

Let an axisymmetrical dock be freely floating on a free surface of water as shown in figure 2.1. $0(x,y,z)$ is a right-handed Cartesian coordinate system fixed in the space. x-and y-axes are taken on the calm water surface, and z-axis is directed vertically upwards. (r,θ,z) refers to the space-fixed cylindrical coordinate system such that

$$x=r\cos\theta$$
$$y=r\sin\theta. \quad (2.1)$$

Let G be the center of mass of the dock, and be on the axis of the dock. At an initial instant of time, the dock is assumed to be in its hydrostatic equilibrium with z-axis coincident with the axis of the dock. Furthermore, the equilibrium is assumed to be stable.

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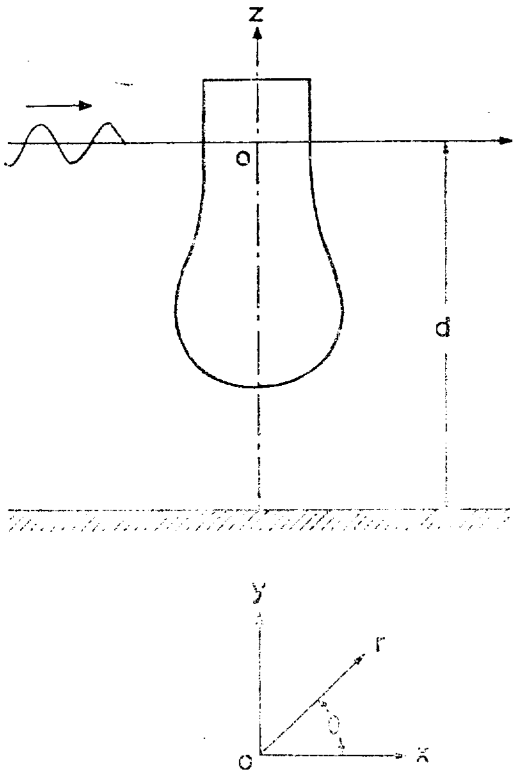

Fig. 2. Coordinate System

Let the free surface elevation $\eta_o(x,y)e^{-i\sigma t}$ of a regular plane wave, which travels in water of the depth d along x-axis from left to right, be approximated by

$$\eta_o(x,y)=\zeta_o e^{ikx}$$
$$=\zeta_o\sum_{m=o}^{\infty}\epsilon_m i_m J_m\ (kr)\ \cos m\theta, \qquad (2.2)$$

where

i=imaginary unit

σ=circular frequency

ζ_o=amplitude of the free surface elevation

$$\epsilon_m=\begin{cases}1 \text{ for } m=0\\ 2 \text{ for } m=1,2,3,\cdots\cdots\end{cases}$$

J_m=Bessel function of the first kind of order m,

and k is the wave number of shallow water:

$$k\ \tanh\ kd=\sigma^2/\mathrm{g}. \qquad (2.3)$$

The velocity potential $\varphi_0(x,y,z)\ e^{-i\sigma t}$ of the regular incident wave (2.2) may be approximated by

$$\varphi_0(x,y,z)=-\zeta_0\frac{ig}{\sigma}\ \frac{\cosh\ k(z+d)}{\cosh\ kd}\ e^{ikx}$$
$$=-\zeta_0\frac{ig}{\sigma}\ \frac{\cosh\ k(z+d)}{\cosh\ kd}$$
$$\times\sum_{m=0}^{\infty}\epsilon_m i^m J_m(kr)\cos m\theta, \qquad (2.4)$$

where g is the gravitational acceleration.

Since G is on the axis of the axisymmetrical dock, the motion of the dock due to the regular plane wave (2.2) is assumed to be confined to x-z plane. If $|\zeta_0|$ is sufficiently small compared with a principal dimension of the dock, the motion of the dock may be assumed to be small from the assumption of the stable hydrostatic equilibrium of the dock. Then, the motion of the fluid also becomes small. The linearized theory is assumed in the followings. Let $X_G e^{-i\sigma t}$, $Z_G e^{-i\sigma t}$ and $\Theta_G e^{-i\sigma t}$ be the stationary harmonic parts of the horizontal, the vertical and the angular displacements of G respectively (figure 2.2), and $\varphi(x,y,z)e^{-i\sigma t}$ be the velocity potential of the fluid motion. Then, the unknowns X_G, Z_G, Θ_G and φ are the solutions of the following linearized boundary value problem (Wehausen & Laitone (1960)) :

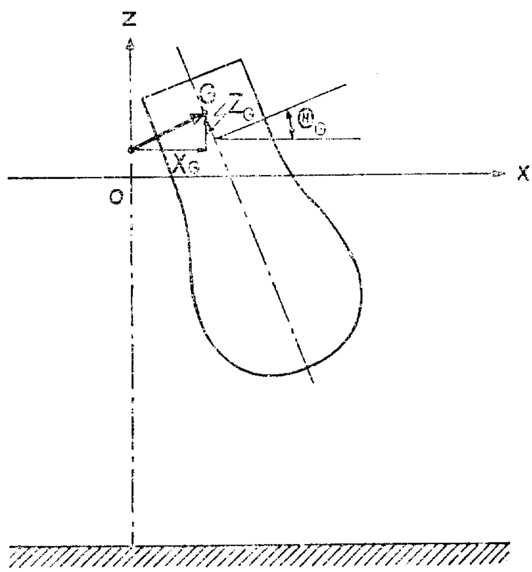

Fig. 2. Definition of displacements.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\varphi=0\ \ in\ D \qquad (2.4a)$$

$$\frac{\partial}{\partial n}\varphi=n_x(-i\sigma X_G)+n_z(-i\sigma Z_G)$$
$$-\{n_x(z-z_G)-n_z x\}(-i\sigma\Theta_G)\ \ on\ S \qquad (2.4b)$$

$$\left(\frac{\partial}{\partial z}-\frac{\sigma^2}{g}\right)\varphi=0 \qquad on\ F_S\ (2.5c)$$

$$\frac{\partial}{\partial z}\varphi=0 \qquad at\ z=-d\ (2.5d)$$

$$\sqrt{r}\left(\frac{\partial}{\partial r}-ik\right)(\varphi-\varphi_0)\sim 0\left(\frac{1}{r}\right) \quad as\ r\to\infty, \tag{2.5e}$$

and

$$-\sigma^2 M X_G=-i\sigma\rho\int_S \varphi n_x\, ds \tag{2.6a}$$

$$(-\sigma^2 M+\rho g S_W)Z_G=-i\sigma\rho\int_S \varphi n_z\, dS \tag{2.6b}$$

$$(-\sigma^2 I_G+\rho g I_z^V+\rho g I_{xx}^{S_W})\Theta_G$$
$$=-i\sigma\rho\int_S\{xn_z-(z-z_G)n_x\}\varphi dS, \tag{2.6c}$$

where

ρ=density of the fluid

D=mean region occupied by the fluid

S=mean wetted surface of the dock

F_S=mean free surface(i.e., calm water surface)

V=mean displaced volume of the dock

S_W=mean water plane area of the dock

$n=(n_x,n_y,n_z)$=unit normal of S into the fluid

Z_G=height of G above calm water surface

M=mass of the dock

I_G=moment of inertia of the dock corresponding to the angular displacement Θ_G

$I_z^V=\int_V(z-Z_G)dV$

$I_{xx}^{S_W}=\int_{S_W}x^2\, dS_W.$

(2.5e) is the Sommerfeld radiation condition.

Since (2.5b) can be written as

$$\frac{\partial}{\partial n}\varphi=n_x(-i\sigma X_G-Z_G i\sigma\Theta_G)+n_z(-i\sigma Z_G)$$
$$+(xn_z-zn_x)(-i\sigma\Theta_G), \tag{2.7}$$

φ may be decomposed as

$$\varphi=\zeta_0\phi_0+\zeta_0\phi_D+(-i\sigma X_G-Z_G i\sigma\Theta_G)\phi_x$$
$$+(-i\sigma Z_G)\phi_z+(-i\sigma\Theta_G)\phi_\theta, \tag{2.8}$$

where

$$\phi_0=-\frac{ig}{\sigma}\frac{\cosh k(z+d)}{\cosh kd}e^{ikx}$$
$$=-\frac{ig}{\sigma}\frac{\cosh k(z+d)}{\cosh kd}\sum_{m=0}^{\infty}\epsilon_m i^m J_m(kr)\cos m\theta, \tag{2.9}$$

and ϕ_D,ϕ_X,ϕ_Z and ϕ_θ are harmonics in D which satisfy the free surface condition, the bottom condition, the Sommerfeld radiation condition and

$$\frac{\partial}{\partial_n}\phi_D=-\frac{\partial}{\partial_n}\phi_0=f_D$$

$$\frac{\partial}{\partial_n}\phi_X=n_x=f_X$$

$$\frac{\partial}{\partial_n}\phi_Z=n_z=f_Z$$

$$\frac{\partial}{\partial_n}\phi_\theta=xn_z-zn_x=_{J\theta}. \quad on\ S \tag{2.10}$$

Substituting (2.8) into (2.6), the following equations are obtained:

$$-\sigma^2 M X_G=-i\sigma\rho\zeta_0 F_{ox}-i\sigma\rho\zeta_0 F_{DX}$$
$$-\sigma^2\rho X_G F_{XX}-\sigma^2\rho\Theta_G(F_{\theta X}+Z_G F_{XX})$$
$$(-\sigma^2 M+\rho g S_W)Z_G=-i\sigma\rho\zeta_0 F_{OZ}-i\sigma\rho\zeta_0 F_{DZ}$$
$$-\sigma^2\rho Z_G F_{ZZ}$$

$$(-\sigma^2 I_G+\rho g I_z^V+\rho g I_{xx}^{S_W})\Theta_G=-i\sigma\rho\zeta_0(F_{O\theta}+Z_G F_{OX})$$
$$-i\sigma\rho\zeta_0(F_{D\theta}+Z_G F_{DX})-\sigma^2\rho X_G(F_{X\theta}+Z_G F_{XX})$$
$$-\sigma^2\rho\Theta_G(F_{\theta\theta}+Z_G F_{X\theta}+Z_G F_{\theta X}+Z_G^2 F_{XX}), \tag{2.11}$$

where

$$F_{IJ}=\int_S\phi_I f_J dk=\int_S\phi_I\frac{\partial\phi_J}{\partial n}ds \tag{2.12}$$
$$(I=O,D,X,Z,\Theta;J=X,Z,\Theta).$$

It must be noticed that ϕ_I and F_{IJ} $(I,J=D,X,Z,\Theta)$ are determined by the geometrical characteristics of S. Because of Green's integral theorem, the hydrodynamic force F_{IJ} $(I,J=D,X,Z,\Theta)$ is equal to F_{JI}:

$$F_{IJ}-F_{JI}=\int_S\left[\phi_I\frac{\partial\phi_J}{\partial n}-\phi_J\frac{\partial\phi_I}{\partial n}\right]dS=0 \tag{2.13}$$
$$(I,J=D,X,Z,\Theta).$$

Let X,Z and Θ be the displacements of the point fixed to the dock which has the mean coordinates $(x=y=z=0)$. It then follows that

$$X_G=X-Z_G\Theta$$
$$Z_G=Z$$
$$\Theta_G=\Theta. \tag{2.14}$$

From (2.10), $-i\sigma X\phi_X$, $-i\sigma Z\phi_Z$ and $-i\sigma\Theta\phi_\theta$ are the velocity potentials due to the simple harmonic oscillations of the dock in still water corresponding to the displacements $X,Z$and Θ respectively. From the pressure equation and (2.12), $-\sigma^2\rho\{X,Z,\Theta\}F_{IJ}$ $(I,J=X,Z,\Theta)$ is the J-th component of the hydrodynamic reaction due to the I-th motion $(I=X,Z,\Theta)$. Let $-\sigma^2\rho\{X_G,Z_G,\Theta_G\}F_{GIJ}$ be the component of the hydrodyamic reaction with respect to the center of mass G corresponding to the displacement (X_G,Z_G,Θ_G). Then, the relations between F_{IJ} and F_{GIJ} can easily be obtained as

$$F_{GXX}=F_{XX}$$
$$F_{GZZ}=F_{ZZ}$$
$$F_{G\theta\theta}=F_{\theta\theta}+z_G F_{\theta X}+z_G F_{X\theta}+z_G^2 F_{XX}$$
$$F_{G\theta X}=F_{\theta X}+z_G F_{XX}$$
$$F_{GX\theta}=F_{X\theta}+z_G F_{XX}, \tag{2.15}$$

where other components are equal to zero from the geometrical and the kinematical symmetricities.

$\mathrm{Re}[-\rho F_{IJ}]$ and $\mathrm{Im}[-\sigma\rho F_{IJ}]$[1] $(I,J=X,Z,\Theta)$ are called the added mass and the damping matrices corresponding to the motion (X,Z,Θ), since

$$\begin{aligned}&\sigma^2\rho\{X,Z,\Theta\}F_{IJ}\\&\quad=\{-\sigma^2X,-\sigma^2Z,-\sigma^2\Theta\}\ \mathrm{Re}\ [-\rho F_{IJ}]\\&\quad+\{-i\sigma X,-i\sigma Z,-i\sigma\Theta\}\ \mathrm{Im}\ [-\sigma\rho F_{IJ}].\end{aligned}\tag{2.16}$$

Let F^*_{IJ} $(I,J=X,Z,\Theta)$ be the complex conjugate of F_{IJ}. It then follows that

$$\begin{aligned}2i\ \mathrm{Im}[F_{IJ}]&=F_{IJ}-F^*_{IJ}\\&=\int_S\left[\phi_I\frac{\partial\phi^*_J}{\partial n}-\phi^*_J\frac{\partial\phi_I}{\partial n}\right]dS\\&=\lim_{r\to\infty}\int_{-d}^{0}dz\int_0^{2\pi}rd\theta\left[\phi_I\frac{\partial\phi^*_J}{\partial r}-\phi^*_J\frac{\partial\phi_I}{\partial r}\right]\end{aligned}\tag{2.17}$$

since, from (2.10), $\partial\phi_I/\partial n$ $(I=X,Z,\Theta)$ is a real valued function on S. Hence, $\mathrm{Im}[F_{IJ}]$ $(I,J=X,Z,\Theta)$ can be determined by the assymptotic behaviour of the corresponding potentials. Let the assymptotic form of ϕ_I $(I=X,Z,\Theta)$, when r tends to infinity, be given as

$$\phi_I\sim-\frac{ig}{\sigma}\ \frac{A_I(\theta)}{\sqrt{kr}}\ \frac{\cosh k(z+d)}{\cosh kd}e^{ikr}\quad r\to\infty.\tag{2.18}$$

Substituting (2.18) into (2.17), $2i\ \mathrm{Im}[F_{IJ}]$ can be derived as

$$\begin{aligned}2i\ \mathrm{Im}[F_{IJ}]&=-\frac{2ig^2}{\sigma^2(\cosh kd)^2}\left(\frac{d}{2}+\frac{\cosh kd\ \sinh kd}{2k.}\right)\\&\cdot\int_0^{2\pi}A_I(\theta)\ A^*_J(\theta)\ d\theta.\end{aligned}\tag{2.19}$$

$-i\sigma\rho\zeta_o(F_{OI}+F_{DI})$ $(I=X,Z,\Theta)$ represents the wave excitation force to the dock fixed in the space, and $-i\sigma\rho\zeta_oF_{OI}$ is called the Froude–Krylov force. From Green's integral theorem, $F_{OI}+F_{DI}$ can be written as

$$\begin{aligned}F_{OI}+F_{EI}&=\int_S(\phi_O+\phi_D)\frac{\partial\phi_I}{\partial n}dS\\&=\int_S\left[\phi_O\frac{\partial\phi_I}{\partial n}-\phi_I\frac{\partial\phi_O}{\partial n}\right]dS\\&=\lim_{r\to\infty}\int_{-d}^{0}dz\int_0^{2\pi}rd\theta\left[\phi_O\frac{\partial\phi_I}{\partial r}-\phi_I\frac{\partial\phi_0}{\partial r}\right]\end{aligned}\tag{2.20}$$

Substitution of (2.9) and (2.18) into (2.20) leads to the following expression of the wave excitation force:

$$\begin{aligned}F_{OI}+F_{DI}&=-\frac{2\sqrt{2\pi}ig^2e^{i\pi/4}}{\sigma^2(\cosh kd)^2}\\&\cdot\left(\frac{d}{2}+\frac{\cosh kd\ \sinh kd}{2k}\right)A_I(\pi).\end{aligned}\tag{2.21}$$

This relation is called the Haskind-Newman relation (Newman (1962)).

3. Variational formulation of the boundary value problem

From discussions in §2, it can be deduced that a boundary value problem of the following type must be solved to determine the wave forces and the hydrodynamic reactions:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=0\ \text{ in } D\tag{3.1a}$$

$$\frac{\partial}{\partial n}\phi=f\ \text{ on } S\tag{3.1b}$$

$$\left(\frac{\partial}{\partial z}-\frac{\sigma^2}{g}\right)\phi=0\qquad\text{on } F_S\tag{3.1c}$$

$$\frac{\partial}{\partial z}\phi=0\qquad\text{at } z=-d\tag{3.1d}$$

$$\sqrt{r}\left(\frac{\partial}{\partial r}-ik\right)\phi\sim 0\left(\frac{1}{r}\right)\qquad\text{as } r\to\infty.\tag{3.1e}$$

Bessho (1968) and Isshiki (1970, 1972) obtained an variational expression for the hydrodynamic force

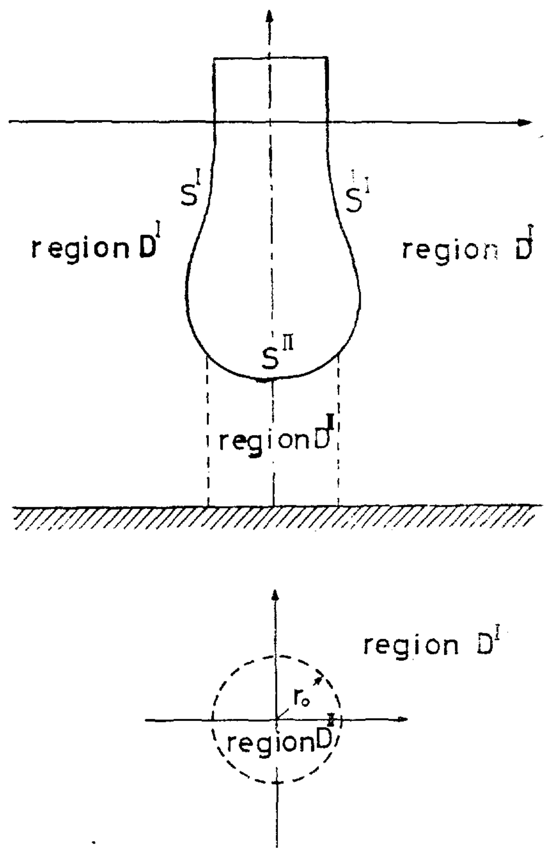

Fig. 3.1. Division of D into subregions D^I and D^{II}.

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1) Re [ ] and Im [ ] mean to take the real and the imaginary parts of the quantity in the square bracket.

associated with the boundary value problem (3.1). In the followings, an extension of their theory will be discussed.

Divide the region D into two subregions D^I and D^{II} as shown in figure 3.1, that is

$$D=D^I+D^{II}.$$

The intersection of S and D^I is denoted by S^I, and that of S and D^{II} by S^{II}. Let ϕ^I and ϕ^{II} refer to the velocity potential ϕ in D^I and D^{II} respectively. Then, the boundary value problem (3.1) may be formulated in terms of ϕ^I and ϕ^{II} as follows:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\begin{Bmatrix}\phi^I\\ \phi_{II}\end{Bmatrix}=0 \qquad \begin{matrix}\text{in } D^I\\ \text{in } D^{II}\end{matrix} \qquad (3.3a)$$

$$\frac{\partial}{\partial n}\begin{Bmatrix}\phi^I\\ \phi_{II}\end{Bmatrix}=f=\begin{matrix}f^I\\ f^{II}\end{matrix} \qquad \begin{matrix}\text{on } S^I\\ \text{on } S^{II}\end{matrix} \qquad (3.3b)$$

$$\left(\frac{\partial}{\partial z}-\frac{\sigma^2}{g}\right)\phi^I=0 \qquad \text{on } Fs \qquad (3.3c)$$

$$\frac{\partial}{\partial z}\begin{Bmatrix}\phi^I\\ \phi_{II}\end{Bmatrix}=0 \qquad \text{at } z=-d \qquad (3.3d)$$

$$\sqrt{r}\left(\frac{\partial}{\partial r}-\mathrm{ik}\right)\phi^I\sim 0\left(\frac{1}{r}\right) \qquad \text{as } r\to\infty, \qquad (3.3e)$$

and the conditions on the intersection of D^I and D^{II}:

$$\phi^I=\phi^{II} \qquad (3.3f)$$

$$\frac{\partial}{\partial r}\phi^I=\frac{\partial}{\partial r}\phi^{II} \qquad \text{on } D^I\cap D^{II}. \qquad (3.3g)$$

The boundary value problem (3.3) can be transformed into a variational problem. Let $L[\phi^I,\ \phi^{II};f]$ be a functional of the argument functions ϕ^I and ϕ^{II} such that

$$L[\phi^I,\ \phi^{II};f]=\sum_{\lambda=I,II}\int_{D^\lambda}\left(\frac{\partial^2\phi^\lambda}{\partial x^2}+\frac{\partial^2\phi^\lambda}{\partial y^2}+\frac{\partial^2\phi^\lambda}{\partial z^2}\right)\phi^\lambda\, dD$$
$$+\sum_{\lambda=I,II}\int_{S^\lambda}\left(\frac{\partial\phi^\lambda}{\partial n}-2f^\lambda\right)\phi^\lambda\, dS$$
$$-\int_{Fs}\left(\frac{\partial\phi^I}{\partial z}-\frac{\sigma^2}{g}\phi^I\right)\phi^I\, dFs$$
$$+\sum_{\lambda=I,II}\int_{(z=-d)^\lambda}\frac{\partial\phi^\lambda}{\partial z}\phi^\lambda\, dxdy$$
$$+\int_{D^I\cap D^{II}}\left(\frac{\partial\phi^I}{\partial r}\phi^I-\frac{\partial\phi^{II}}{\partial r}\phi^{II}\right)d(D^I\cap D^{II}). \qquad (3.4)$$

It then follows that

$$\delta L=2\sum_{\lambda=I,II}\int_{D^\lambda}\left(\frac{\partial^2\phi^\lambda}{\partial x^2}+\frac{\partial^2\phi^\lambda}{\partial y^2}+\frac{\partial^2\phi^\lambda}{\partial z^2}\right)\delta\phi^\lambda\, dD$$
$$+2\sum_{\lambda=I,II}\int_{S^\lambda}\left(\frac{\partial\phi^\lambda}{\partial n}-f^\lambda\right)\delta\phi^\lambda\, dS$$
$$-2\int_{F_s}\left(\frac{\partial\phi^I}{\partial z}-\frac{\sigma^2}{g}\phi^I\right)\delta\phi^I\, dFs$$
$$+2\sum_{\lambda=I,II}\int_{(z=-d)^\lambda}\frac{\partial\phi^\lambda}{\partial z}\delta\phi^\lambda\, dxdy$$
$$-\lim_{r\to\infty}\int_{-d}^{0}dz\int_0^{2\pi}rd\theta\left(\frac{\partial\phi^I}{\partial r}\delta\phi^I-\frac{\partial\delta\phi^I}{\partial r}\phi^I\right)$$
$$+2\int_{D^I\cap D^{II}}\left(\frac{\partial\phi^I}{\partial r}\delta\phi^I-\frac{\partial\phi^{II}}{r\delta}\phi^{II}\right)d(D^I\cap D^{II}). \qquad (3.5)$$

It is easily seen that (3.3a-d) and (3.3g) are the stationary conditions of the functional L under the subsidiary conditions (3.3e) and (3.3f). Hence, the following variational problem is equivalent to the boundary value problem (3.3):

$$L[\phi^I,\phi^{II};f]=\text{stationary}, \qquad (3.6a)$$

under

$$\sqrt{r}\left(\frac{\partial}{\partial r}-ik\right)\phi^I\sim 0\left(\frac{1}{r}\right) \quad \text{as } r\to\infty \qquad (3.6b)$$

$$\phi^I=\phi^{II} \quad on\ D^I\cap D^{II}. \qquad (3.6c)$$

It must be noticed that $L[\phi^I,\phi^{II};f]$ is a variational expression for a quantity$-\int_S\phi f\ dS$, that is

$$L[\phi^I,\phi^{II};f]=-\sum_{\lambda=I,II}\int_{S^\lambda}\phi^\lambda f^\lambda\ dS, \qquad (3.7)$$

When ϕ^λ (λ=I,II) is the solution of the variational problem (3.6).

If the natural conditions (3.3a), (3.3c) and (3.3d) of the variational problem (3.6) are also considered as the subsidiary ones, the following variational problem is then obtained:

$$M[\phi^I,\phi^{II};f]=L[\phi^I,\phi^{II};f]$$
$$=\sum_{\lambda=I,II}\int_{S^\lambda}\left(\frac{\partial\phi^\lambda}{\partial n}-2f^\lambda\right)\phi^\lambda\ dS$$
$$+\int_{D^I\cap D^{II}}\left(\frac{\partial\phi^I}{\partial r}\phi^I-\frac{\partial\phi^{II}}{\partial r}\phi^{II}\right)d(D^I\cap D^{II})$$
$$=\text{stationary}, \qquad (3.8a)$$

under

$$\left(\frac{\partial^2}{\partial_x^2}+\frac{\partial^2}{\partial_y^2}+\frac{\partial^2}{\partial_z^2}\right)\begin{Bmatrix}\phi^I\\ \phi^{II}\end{Bmatrix}=0 \quad \begin{matrix}\text{in } D^I\\ \text{in } D^{II}\end{matrix} \qquad (3.8b)$$

$$\left(\frac{\partial}{\partial z}-\frac{\sigma^2}{g}\right)\phi^I=0 \qquad \text{on } Fs \qquad (3.8c)$$

$$\frac{\partial}{\partial z}\begin{Bmatrix}\phi^I\\ \phi^{II}\end{Bmatrix}=0 \qquad \text{at } z=-d \qquad (3.8d)$$

$$\sqrt{r}\left(\frac{\partial}{\partial r}-ik\right)\phi^I\sim 0\left(\frac{1}{r}\right) \quad \text{as } r\to\infty, \qquad (3.8e)$$

and

$$\phi^I=\phi^{II} \qquad \text{on } D^I\cap D^{II}. \qquad (3.8f)$$

The natural conditions of this variational problem are as follows:

$$\frac{\partial}{\partial n}\begin{Bmatrix}\phi^I\\ \phi^{II}\end{Bmatrix}=\begin{Bmatrix}f^I\\ f^{II}\end{Bmatrix} \qquad \begin{matrix}\text{on } S^I\\ \text{on } S^{II}\end{matrix} \qquad (3.8g)$$

$$\frac{\partial}{\partial r}\phi^I=\frac{\partial}{\partial r}\phi^{II} \qquad \text{on } D^I\cap D^{II}. \qquad (3.8h)$$

The variational problem (3.8) is an extension of the Bessho variational formulation (Bessho (1968)).

If the continuity of the normal velecity on $D^{\text{I}}\cap D^{\text{II}}$ is assumed, a complementary variational formulation of (3.6) can be obtained as follows:

$$L^*[\phi^{\text{I}},\phi^{\text{II}};f]=L[\phi^{\text{I}},\phi^{\text{II}};f]$$
$$-2\int_{L^{\text{I}}\cap D^{\text{II}}}\left(\frac{\partial\phi^{\text{I}}}{\partial r}\phi^{\text{I}}-\frac{\partial\phi^{\text{II}}}{\partial r}\phi^{\text{II}}\right)d(D^{\text{I}}\cap D^{\text{II}})$$
$$=\text{stationary}, \tag{3.9}$$

under (3.3e) and (3.3g). (3.3a-d) and (3.3f) then become the natural conditions of this variational problem.

Corresponding to (3.8), it then follows the following complementary formulation:

$$M^*[\phi^{\text{I}},\phi^{\text{II}};f]=M[\phi^{\text{I}},\phi^{\text{II}};f]$$
$$-2\int_{D^{\text{I}}\cap D^{\text{II}}}\left(\frac{\partial\phi^{\text{I}}}{\partial r}\phi^{\text{I}}-\frac{\partial\phi^{\text{II}}}{\partial r}\phi^{\text{II}}\right)d(D^{\text{I}}\cap D^{\text{II}})$$
$$=\text{stationary}, \tag{3.10}$$

under (3.3a), (3.3c-e) and (3.3g). Then, (3.3b) and (3.3f) are given as the natural conditions of (3.10).

4. Rayleigh-Ritz procedure for the variational formulation

An admissible function for the variational problem (3.8) will be obtained in the followings, and a Rayleigh-Ritz approximation based on the admissible function will be discussed.

Let ϕ:

$$\phi=\sum_{m=0}^{\infty}\epsilon_m i^m\Psi_m(r,z)\cos m\theta$$
$$=\begin{cases}\phi^{\text{I}}\\ \phi^{\text{II}}\end{cases}=\sum_{m=0}^{\infty}\epsilon_m i^m\begin{cases}\Psi^{\text{I}}_m(r,z) & \\ \Psi^{\text{II}}_m(r,z) & \end{cases}\cos m\theta\quad\begin{matrix}\text{in } D^{\text{I}}\\ \text{in } D^{\text{II}}\end{matrix} \tag{4.1}$$

be a solution of the Laplace equation:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)\phi=0. \tag{4.2}$$

It then follows that

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}-\frac{m^2}{r^2}\right)\Psi_m=0. \tag{4.3}$$

The solution of this equation by the separation of the variables are as follows:

$$\left.\begin{matrix}K_m(\alpha r)\\ I_m(\alpha r)\end{matrix}\right\}\times\begin{cases}\cos\alpha z & J_m(\beta r)\\ \sin\alpha z, & H^{(1)}_m(\beta r)\end{cases}\times\begin{cases}e^{\beta z}\\ e^{-\beta z},\end{cases} \tag{4.4}$$

where

α,β=real parameters

I_m, K_m=modified Bessel functions of order m

$H^{(1)}_m$=Hankel function of the first kind of order m.

$I_m(\alpha r)$, $J_m(\beta r)$ are regular at $r=0$, and $K_m(\alpha r)$, $H^{(1)}_m(\beta r)$ unbounded at $r=0$. When r tends to infinity, these functions have the following assymptotic expansions:

$$K_m(\alpha r)\sim\sqrt{\frac{\pi}{2\alpha r}}e^{-\alpha r}$$
$$I_m(\alpha r)\sim\frac{1}{\sqrt{2\pi\alpha r}}\left[e^{\alpha r}+e^{-\alpha r-(m+1/2)\pi i}\right]$$
$$J_m(\beta r)\sim\frac{1}{\sqrt{2\pi\beta r}}[e^{i\{\beta r-\pi(2m+1)/4\}}$$
$$+e^{-\{\beta r-\pi(2m+1)/4\}}] \tag{4.5}$$
$$H^{(1)}_m(\beta r)\sim\sqrt{\frac{2}{\pi\beta r}}e^{i\{\beta r-\pi(2m+1)/4\}}.$$

From the properties at $r=0,\infty$ and the condition at $z=-d$, it may be assumed as follows:

$$\left.\begin{matrix}\Psi^{\text{I}}_m(r,z)\\ \Psi^{\text{II}}_m(r,z)\end{matrix}\right\}=\text{linear combination of}$$
$$\begin{cases}K_m(\alpha r)\cos\alpha(z+d)\\ I_m(\alpha' r)\cos\alpha'(z+d)\end{cases}\text{and}$$
$$\begin{cases}H^{(1)}_m(\beta r)\cosh\beta(z+d)\\ J_m(\beta' r)\cosh\beta'(z+d).\end{cases} \tag{4.6}$$

The free surface condition (3.3c) then yields the following eigen-value equations for α and β:

$$\alpha\tan\alpha d=-\sigma^2/g\quad \alpha>0 \tag{4.7a}$$
$$\beta\tanh\beta d=\sigma^2/g\quad \beta>0. \tag{4.7b}$$

(4.7a) has an infinite number of roots $\alpha_1<\alpha_2<\alpha_3<\cdots$ and the solution of (4.7b) is equal to the wave number of shallow water k (see (2.3)). Hence, $\Psi^{\text{I}}_m(r,z)$ may be assumed as

$$\Psi^{\text{I}}_m(r,z)=\sum_{i=1}^{\infty}A^{\text{I}}_{mi}\frac{K_m(\alpha_i r)}{K_m(\alpha_i r_0)}\cos\alpha_i(z+d)$$
$$+B^{\text{I}}_m H^{(1)}_m(kr)\frac{\cosh k(z+d)}{\cosh kd}, \tag{4.8}$$

where A^{I}_{mi} $(i=1,2,\cdots\cdots)$ and B^{I}_m are integral constants, and r_0 is the radius of the cylinder D^{II}(figure 3.1).

From (3.8f), $\Psi^{\text{II}}_m(r,z)$ must be a function such that

$$\Psi^{\text{II}}_m(r_0,z)=\Psi^{\text{I}}_m(r_0,z)\qquad \text{on} D^{\text{I}}\cap D^{\text{II}}. \tag{4.9}$$

Let $\Psi^{\text{II}}_m(r,z)$ be decomposed as

$$\Psi^{\text{II}}_m(r,z)=\Psi^{\text{II}'}_m(r,z)+\Psi^{\text{II}''}_m(r,z), \tag{4.10a}$$

where

$$\Psi_m^{II'}(r_0,z)=\Psi_m^{I}(r_0,z) \qquad (4.10b)$$
on $D^I \cap D^{II}$.
$$\Psi_m^{II''}(r_0,z)=0 \qquad (4.10c)$$

Then, it may be assumed that

$$\Psi_m^{II'}(r,z)=\sum_{i=1}^{\infty}A_{mi}^{II}\frac{I_m(\alpha_i r)}{I_m(\alpha_i r_0)}\cos\alpha_i(z+d) + B_m^{II}J_m(kr)\frac{\cosh k(z+d)}{\cosh kd,} \qquad (4.11a)$$

where

$$A_{mi}^{II}=A_{mi}^{I} \qquad (i=1,2,\cdots\cdots) \qquad (4.11b)$$

$$B_m^{II}J_m(kr_0)=B_m^{I}H_m^{(1)}(kr_0). \qquad (4.11c)$$

Let $j_{m,i}$ be the i-th zero point of the Bessel function J_m:

$$J_m(j_{m,i})=0 \qquad 0<j_{m,1}<j_{m,2}<\cdots\cdots \qquad (4.12)$$

Then, $\Psi_m^{II''}(r,z)$ may be written as

$$\Psi_m^{II''}(r,z)=\sum_{i=1}^{\infty}B_{mi}^{II}J_m(k_{mi}r)\frac{\cosh k_{mi}(z+d)}{\cosh k_{mi}d} \qquad (4.13a)$$

where B_{mi}^{II} $(i=1,2,\cdots\cdots)$ are integral constants, and

$$k_{mi}=j_{m,i}/r_0 \qquad (4.13b)$$

Hence, $\Psi_m^{II}(r,z)$ may be assumed as

$$\Psi_m^{II}(r,z)=\sum_{i=1}^{\infty}A_{mi}^{I}\frac{I_m(\alpha_i r)}{I_m(\alpha_i r_0)}\cos\alpha_i(z+d) + B_m^{I}\frac{H_m^{(1)}(kr_0)}{J_m(kr_0)}J_m(kr)\frac{\cosh k(z+d)}{\cosh kd} + \sum_{i=1}^{\infty}B_{mi}^{II}J_m(k_{mi}r)\frac{\cosh k_{mi}(z+d)}{\cosh k_{mi}d} \qquad (4.14)$$

In the followings, an approximation based on the variational problem (3.8) and the admissible functions (4.8) and (4.14) will be considered. For the sake of simplicity, a numerical procedure will be discussed for a circular dock of the radius a (figure 4.1). And r_0 is assumed to be equal to a.

Let f be expanded on S as

$$f=\begin{cases} f^{I}(z,\theta) \\ f^{II}(r,\theta) \end{cases} = \sum_{m=0}^{\infty}\epsilon_m\, i^m \begin{cases} f_m^{I}(z) \\ f_m^{II}(r) \end{cases}\cos m\theta \qquad \begin{matrix}\text{on } S^I \\ \text{on } S^{II}.\end{matrix} \qquad (4.15)$$

Substituting (4.1) and (4.15) into the functional $M[\phi^I,\phi^{II};f]$:

$$M[\phi^I,\phi^{II};f]=\int_{-(d-h)}^{0}\int_0^{2\pi}\left[\frac{\partial\phi^I}{\partial r}-2f^I(z,\theta)\phi^I\right]_{r=a}ad\theta dz + \int_0^a\int_0^{2\pi}\left[\left(-\frac{\partial\phi^{II}}{\partial z}-2f^{II}(r,\theta)\right)\phi^{II}\right]_{z=-(d-h)}rdrd\theta + \int_{-d}^{-(d-h)}\int_0^{2\pi}\left[\frac{\partial\phi^I}{\partial r}\phi^I-\frac{\partial\phi^{II}}{\partial r}\phi^{II}\right]_{r=a}ad\theta dz, \qquad (4.16)$$

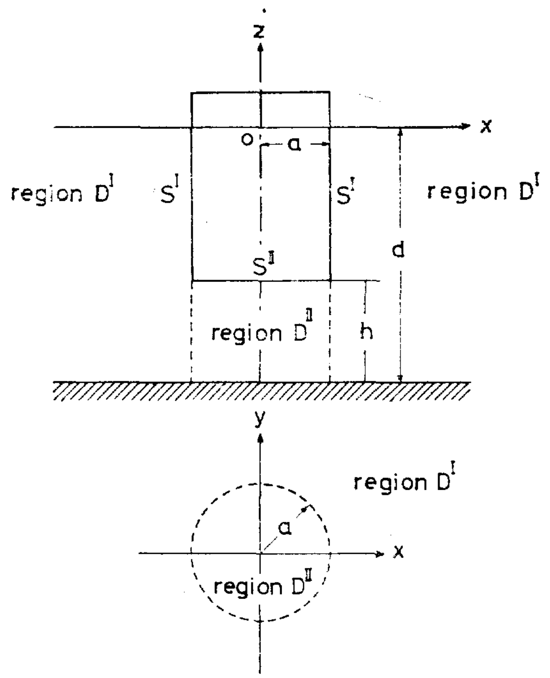

Fig. 4. Circular dock.

the following expression for M is obtained:

$$M[\phi^I,\phi^{II};f]=\sum_{m=0}^{\infty}\epsilon_m(-1)^m M_m[\Psi_m^I,\Psi_m^{II};f_m^I,f_m^{II}], \qquad (4.17a)$$

where

$$\frac{1}{2\pi}M_m\left[\Psi_m^I,\Psi_m^{II};f_m^I,f_m^{II}\right] = a\int_{-(d-h)}^{0}\left[\left(\frac{\partial\Psi_m^I}{\partial r}-2f_m^I(z)\right)\Psi_m^I\right]_{r=a}dz - \int_0^a\left[\left(\frac{\partial\Psi_m^{II}}{\partial z}+2f_m^{II}(r)\right)\Psi_m^{II}\right]_{z=-(d-h)}rdr + a\int_{-d}^{-(d-h)}\left[\frac{\partial\Psi_m^I}{\partial r}\Psi_m^I\right]_{r=a}dz - a\int_{-d}^{-(d-h)}\left[\frac{\partial\Psi_m^{II}}{\partial r}\Psi_m^{II}\right]_{r=a}dz. \qquad (4.17b)$$

Substitution of (4.8) and (4.14) into (4.17b) leads to the following expression of M_m:

$$\frac{1}{2\pi}M_m\left[\Psi^{\rm I}_m,\Psi^{\rm II}_m;f^{\rm I}_m,f^{\rm II}_m\right]$$
$$=\frac{1}{2\pi}M_m\left[A^{\rm I}_{mi},B^{\rm I}_m,B^{\rm II}_{mi};f^{\rm I}_m,f^{\rm II}_m\right]$$
$$=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[A^{\rm I}_m,A^{\rm I}_m\right]_{ij}A^{\rm I}_{mi}A^{\rm I}_{mj}$$
$$+\sum_{i=1}^{\infty}\left[A^{\rm I}_m,B^{\rm I}_m\right]_iA^{\rm I}_{mi}B^{\rm I}_m$$
$$+\left[B^{\rm I}_m,B^{\rm I}_m\right]\left(B^{\rm I}_m\right)^2+\sum_{i=1}^{\infty}\left[B^{\rm I}_m,B^{\rm II}_m\right]_iB^{\rm I}_mB^{\rm II}_{mi}$$
$$+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[A^{\rm I}_m,B^{\rm II}_m\right]_{ij}A^{\rm I}_{mj}B^{\rm II}_{mj}$$
$$+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[B^{\rm II}_m,B^{\rm II}_m\right]_{ij}B^{\rm II}_{mi}B^{\rm II}_{mj}$$
$$-2\sum_{i=1}^{\infty}\left[A^{\rm I}_m\right]_iA^{\rm I}_{mi}-2\left[B^{\rm I}_m\right]B^{\rm I}_m$$
$$-2\sum_{i=1}^{\infty}\left[B^{\rm II}_m\right]_iB^{\rm II}_{mi},\qquad(4.18)$$

where $\left[A^{\rm I}_m,A^{\rm I}_m\right]_{ij}$, etc. are given in appendix A. Because of Green's integral theorem, the following relations hold:

$$\left[A^{\rm I}_m,A^{\rm I}_m\right]_{ij}=\left[A^{\rm I}_m,A^{\rm I}_m\right]_{ji}$$
$$\left[B^{\rm II}_m,B^{\rm II}_m\right]_{ij}=\left[B^{\rm II}_m,B^{\rm II}_m\right]_{ji}.\qquad(4.19)$$

From (4.17a) and (4.18), the stationary condition of the functional M is given as

$$M_m\left[A^{\rm I}_{mi},B^{\rm I}_m,B^{\rm II}_{mi};f^{\rm I}_m,f^{\rm II}_m\right]$$
$$=\text{stationary}\qquad(4.20)$$

for $m=0,1,2,\cdots$. Hence, the following linear algebraic equation for the unknowns $A^{\rm I}_{mi}$, $B^{\rm I}_m$ and $B^{\rm II}_{mi}$ is obtained:

$$\sum_{j=1}^{\infty}2\left[A^{\rm I}_m,A^{\rm I}_m\right]_{ij}A^{\rm I}_{mj}+\left[A^{\rm I}_m,B^{\rm I}_m\right]_iB^{\rm I}_m$$
$$+\sum_{j=1}^{\infty}\left[A^{\rm I}_m,B^{\rm II}_m\right]_{ij}B^{\rm II}_{mj}=2\left[A^{\rm I}_m\right]_i$$
$$\sum_{j=1}^{\infty}\left[A^{\rm I}_m,B^{\rm I}_m\right]_jA^{\rm I}_{mj}+2\left[B^{\rm I}_m,B^{\rm I}_m\right]B^{\rm I}_m$$
$$+\sum_{j=1}^{\infty}\left[B^{\rm I}_m,B^{\rm II}_m\right]_jB^{\rm II}_{mj}=2\left[B^{\rm I}_m\right]$$
$$\sum_{j=1}^{\infty}\left[A^{\rm I}_m,B^{\rm II}_m\right]_{ji}A^{\rm I}_{mj}+\left[B^{\rm I}_m,B^{\rm II}_m\right]_iB^{\rm I}_m$$
$$+\sum_{j=1}^{\infty}2\left[B^{\rm II}_m,B^{\rm II}_m\right]_{ij}B^{\rm II}_{mj}=2\left[B^{\rm II}_m\right]_i\qquad(4.21)$$

where $i=1,2,\cdots$. $\left[A^{\rm I}_m\right]_i$, $\left[B^{\rm I}_m\right]$ and $\left[B^{\rm II}_m\right]_i$ are determined according to the conditions on S. These conditions are given in table 4.1, where "heave", "surge" and "pitch" refer to the vertical (Z) motion, the horizontal (X) motion and the angular (Θ) motion about y-axis respectively ((2.10)). By using the formulae in appendices A-B and table 4.1, $\left[A^{\rm I}_m\right]_i$ $\left[B^{\rm I}_m\right]$ and $\left[B^{\rm II}_m\right]_i$ for these motions can be written explicitely as shown in table 4.2. In numerical calculations, the infinite series in (4.21) are truncated by finite terms according to an approximation:

$$A^{\rm I}_{mi}=0\qquad\text{for } i>p$$
$$B^{\rm II}_{mi}=0\qquad\text{for } i>Q\qquad(4.22)$$

where P and Q are appropriate integers determined by the numerical convergence.

The added mass and damping can be approximated as follows. From (2.12), (2.16) and (3.8), these quantities can be written as

$$\left.\begin{matrix}\text{added mass}\\ \text{damping}\end{matrix}\right\}=\begin{matrix}-\rho\ Re\\ -\sigma\rho\ Im\end{matrix}\Bigg\}\left[\int s^{\rm I}\phi^{\rm I}f^{\rm I}dS\right.$$
$$\left.+\int s^{\rm II}\phi^{\rm II}f^{\rm II}dS\right]=\begin{matrix}\rho\ Re\\ \sigma\rho\ Im\end{matrix}\Bigg\}[M[\phi^{\rm I},\phi^{\rm II};f]],\qquad(4.23a)$$

Table 4.1. Kinematical condition on S.

	$f^{\rm I}(z,\theta)$	$f^{\rm II}(r,\theta)$	m	$f^{\rm I}_m(z)$	$f^{\rm II}_m(r)$
heave(Z)	0	-1	0	0	-1
surge(X)	$\cos\theta$	0	1	$-i/2$	0
pitch(Θ)	$-z\cos\theta$	$-r\cos\theta$	1	$iz/2$	$ir/2$

Table 4.2. $\left[A_m^{\rm I}\right]_i$, $\left[B_m^{\rm I}\right]$ and $\left[B_m^{\rm II}\right]_i$

	heave(Z); $m=0$	surge(X); $m=1$
$\left[A_m^{\rm I}\right]_i$	$-\frac{a}{\alpha_i}\frac{I_1(\alpha_i a)}{I_0(\alpha_i a)}\cos\alpha_i h$	$-\frac{ia}{2\alpha_i}(\sin\alpha_i d-\sin\alpha_i h)$
$\left[B_m^{\rm I}\right]$	$-\frac{a}{k}J_1(ka)\frac{\cosh kh}{\cosh kd}\frac{H_0^{(1)}(ka)}{J_0(ka)}$	$-\frac{ia}{2k}H_1^{(1)}(ka)\left(\frac{\sinh kd}{\cosh kd}-\frac{\sinh kh}{\cosh kd}\right)$
$\left[B_m^{\rm II}\right]_i$	$-\frac{a}{k_{0i}}J_1(k_{0i}a)\frac{\cosh k_{0i}h}{\cosh k_{0i}d}$	0
	pitch(Θ); $m=1$	
$\left[A_m^{\rm I}\right]_i$	$i\frac{a}{2}\left(\frac{d}{\alpha_i}\sin\alpha_i d+\frac{\cos\alpha_i d}{\alpha_i^2}-\frac{h}{\alpha_i}\sin\alpha_i h-\frac{\cos\alpha_i h}{\alpha_i^2}\right)+\frac{i}{2}\frac{a^2}{\alpha_i}\frac{I_2(\alpha_i a)}{I_1(\alpha_i a)}\cos\alpha_i h$	
$\left[B_m^{\rm I}\right]$	$i\frac{a}{2}H_1^{(1)}(ka)\left(\frac{d}{k}\frac{\sinh kd}{\cosh kd}-\frac{1}{k^2}-\frac{h}{k}\frac{\sinh kh}{\cosh kd}+\frac{1}{k^2}\frac{\cosh kh}{\cosh kd}\right)+\frac{i}{2}\frac{a^2}{k}J_2(ka)\frac{\cosh kh}{\cosh kd}\frac{H_1^{(1)}(ka)}{J_1(ka)}$	
$\left[B_m^{\rm II}\right]_i$	$\frac{ia^2}{2k_{1i}}J_2(k_{1i}a)\frac{\cosh k_{1i}h}{\cosh k_{1i}d}$	

where $(\phi^{\rm I},\phi^{\rm II})$ is the exact solution of the variational problem (3.8). From (4.17a), it then follows that

$$\left.\begin{matrix}\text{added mass}\\ \text{damping}\end{matrix}\right\}=\left.\begin{matrix}\rho\ Re\\ \sigma\rho\ I_m\end{matrix}\right\}$$

$$\left[\epsilon_m(-1)^m M_m\left[\psi_m^{\rm I},\ \psi_m^{\rm II};f_m^{\rm I},\ f_m^{\rm II}\right]\right] \quad (4.23b)$$

where

$m=0$ for heave

$m=1$ for surge or pitch.

Let $A_{mi}^{\rm I\cdot}(i=1,2,\cdots\cdots P)$, $B_m^{\rm I\cdot}$ and $B_{m_i}^{\rm II\cdot}(i=1,2,\cdots\cdots Q)$ be an approximate solution of (4.21) based on the assumption (4.22), and $(\psi_m^{\rm I\cdot},\ \psi_m^{\rm II\cdot})$ the corresponding approximation of $(\psi_m^{\rm I},\ \psi_m^{\rm II})$ Then, $M_m\left[\psi_m^{\rm I},\ \psi_m^{\rm II};f_m^{\rm I}\ f_m^{\rm II}\right]$ may be approximated by $M_m\left[\psi_m^{\rm I\cdot},\ \psi_m^{\rm II\cdot};\ f_m^{\rm I},\ f_m^{\rm II}\right]$ that is

$$M_m\left[\psi_m^{\rm I},\psi_m^{\rm II};f_m^{\rm I},\ f_m^{\rm II}\right]\doteqdot M_m\left[\psi_m^{\rm I\cdot},\psi_m^{\rm II\cdot};f_m^{\rm I},f_m^{\rm II}\right] \quad (4.24)$$

From (4.18) and (4.20), the following expression for $M_m\left[\psi_m^{\rm I\cdot},\ \psi_m^{\rm II\cdot};\ f_m^{\rm I},\ f_m^{\rm II}\right]$ is obtained:

$$\frac{1}{2\pi}M_m\left[\psi_m^{\rm I\cdot},\ \psi_m^{\rm II\cdot};\ f_m^{\rm I},\ f_m^{\rm II}\right]$$

$$=\frac{1}{2\pi}M_m\left[A_{mi}^{\rm I\cdot},\ B_m^{\rm I},\ B_{mi}^{\rm II\cdot};\ f_m^{\rm I},\ f_m^{\rm II}\right]$$

$$=-\sum_{i=1}^{P}\left[A_m^{\rm I}\right]_i A_{mi}^{\rm I\cdot}-\left[B_m^{\rm I\cdot}\right]B_m^{\rm I\cdot}$$

$$-\sum_{i=1}^{Q}\left[B_m^{\rm II}\right]_i B_{mi}^{\rm II\cdot} \quad (4.25)$$

The coupling force such as the surge (pitch) component in pitch (surge) is also easily obtained (Isshiki (1970)). If the numerical convergence is satisfactory, the radiation wave amplitude may be calculated from $B_m^{\rm I\cdot}$ by using the assymptotic expansion of $H_m^{(1)}(kr)$ ((4.5)) and the linearized dynamical condition on the free surface:

$$\frac{\partial}{\partial t}(\phi e^{-i\sigma t})+g\eta e^{-i\sigma t}=0\ \ at\ \ z=0 \quad (4.26)$$

where $\eta e^{-i\sigma t}$ is the free surface elevation. From the Haskind-Newman relation (2.21), the wave excitation force will be easily calculated. [1)]

5. Numerical calculations and discussions

The added mass and damping of a circular dock (figure 4.1) were calculated on the basis of the numerical procedure developed in §4. The numerical calculations were carried out by the computer IBM 1130 at Seoul National University, and the single precision (six significant figures) were adopted.

The added mass and the damping coefficients are defined as follows:

1) From a standpoint of the variational calculus, the radiation wave amplitude or the wave excitation force should be approximated by using the radiation and the scattering potentials (Isshiki (1970)).

addedmass coefft. = added mass/$[\rho\pi a^2(d-h)]$
damping coefft. = damping/$[\sigma\rho\pi a^2(d-h)]$.

(5.1)

In tables 5.1—2, the numerical tendancies of the added mass and the damping coeffcients with increasing P and Q are shown, and seem to be satisfactory. From tables 5.3—4, the convergence of B_m^1seems also to be reasonable.

Table 5.1. Convergence of the added mass and the damping coefficients (heave); ($\sigma^2a/g=1$, $d/a=6$, $h/a=4$).

P \ Q		1	2
5	added mass coefft.	0.2462	0.2609
	damping coefft.	0.00422	0.00393
6	added mass coefft.	0.2519	0.2665
	damping coefft.	0.00412	0.00393

Table 5.2. Convergence of the added mass and the damping coefficients (surge); ($\sigma^2a/g=1$, $d/a=6$, $h/a=4$).

P \ Q		1	2
5	added mass coefft.	0.5411	0.5415
	damping coefft.	0.5346	0.5351
6	added mass coefft.	0.5429	0.5433
	damping coefft.	0.5354	0.5359

Table 5.3. Convergence of $B_0^1/\cosh kd$ (heave); ($\sigma^2a/g=1$, $d/a=6$, $h/a=4$).

P \ Q	1	2
5	$(0.2821+0.4959i)\cdot 10^{-3}$	$(0.2720+0.4786i)\cdot 10^{-3}$
6	$(0.2787+0.4900i)\cdot 10^{-3}$	$(0.2686+0.4727i)\cdot 10^{-3}$

Table 5.4. Converergence of $B_1^1/\cosh kd$ (surge); ($\sigma^2a/g=1$, $d/a=6$, $h/a=4$).

P \ Q	1	2
5	$-(0.4273+0.1541i)\cdot 10^{-2}$	$-(0.4275+0.1543i)\cdot 10^{-2}$
6	$-(0.4276+0.1543i)\cdot 10^{-2}$	$-(0.4277+0.1545i)\cdot 10^{-2}$

In figures 5.1—2, the frequency dependencies of the added mass and the damping coefficients are

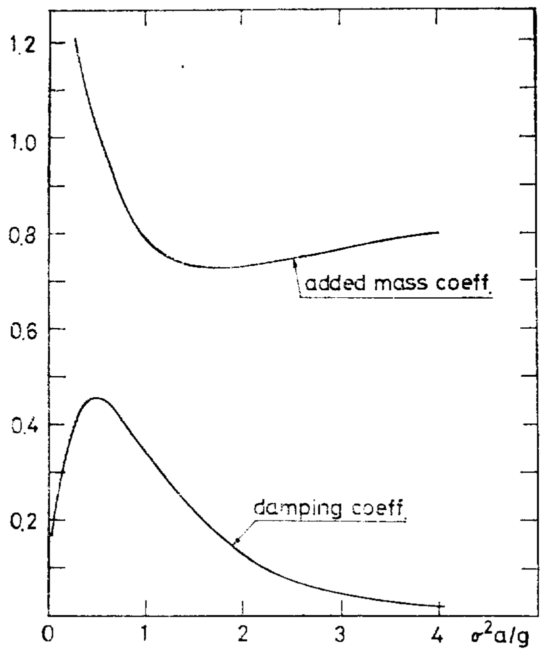

Fig. 5.1 Added mass and damping. coeff. for a circular dock in heave (Z). $d/a=6$, $h/a=5.5$ ($P=6$, $Q=1$)

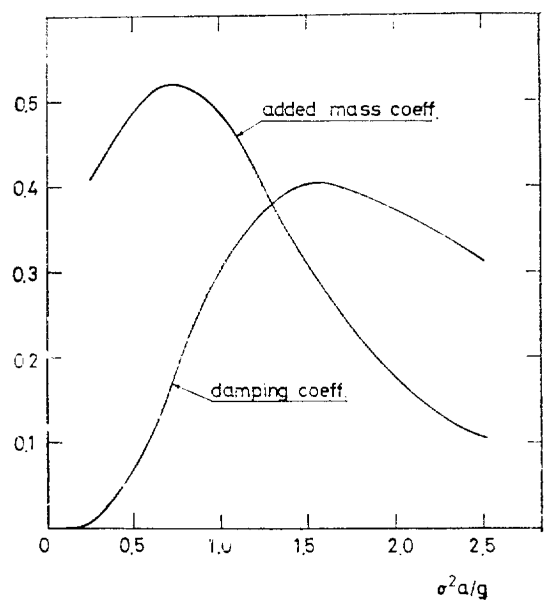

Fig. 5.2. Added massrand Edamping cofef for a circular dock in surge (X) $d/a=6$, $h/a=5.5$ ($P=6, Q=1$).

shown. The free surface effects seem to be compatible with the results of a semi-sphere by Havelock (1955) Barakat (1962) and spheroids by Kim (1965), Sao, Maeda & Hwang (1971). The horizontal force on the dock (d/a=0.75, h/a=0.25, 0.5) due to wave (k= 1.32) was compared with the corresponding results by Garrett (1971), and the agreement was satisfactory.

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Appendix A.

$$\left[A^{\mathrm{I}}_m, A^{\mathrm{I}}_m \right]_{ij} = \alpha_i \, a \frac{K'_m(\alpha_i a)}{K_m(\alpha_i a)} \int_{-d}^{0} \cos \alpha_i (z+d) \cos \alpha_j (z+d) \, dz$$

$$+ \frac{\alpha_i}{I_m(\alpha_i a) I_m(\alpha_j a)} \int_0^a r \, I_m(\alpha_i r) \, I_m(\alpha_j r) \, dr \, \sin \alpha_i h \cos \alpha_j h$$

$$- \alpha_i \, a \frac{I'_m(\alpha_i a)}{I_m(\alpha_i a)} \int_{-d}^{-(d-h)} \cos \alpha_i (z+d) \cos \alpha_j (z+d) \, dz$$

$$\left[A^{\mathrm{I}}_m, B^{\mathrm{I}}_m \right]_i = \left\{ ka \, H^{(1)\prime}_m(ka) + \alpha_i a \, H^{(1)}_m(ka) \frac{K'_m(\alpha_i a)}{K_m(\alpha_i a)} \right\} \frac{1}{\cosh kd} \cdot \int_{-d}^{0} \cosh k(z+d) \cos \alpha_i (z+d) \, dz$$

$$+ \left\{ - \frac{1}{I_m(\alpha_i a)} \int_0^a r \, J_m(kr) \, I_m(\alpha_i r) \, dr \left(k \frac{\sinh kh}{\cosh kd} \cos \alpha_i h - \alpha_i \frac{\cosh kh}{\cosh kd} \sin \alpha_i h \right) \right.$$

$$\left. - \left(ka \, J'_m(ka) + \alpha_i a J_m(ka) \frac{I'_m(\alpha_i a)}{I_m(\alpha_i a)} \right) \cdot \frac{1}{\cosh kd} \int_{-d}^{-(d-h)} \cosh k(z+d) \cos \alpha_i (z+d) dz \right\} \frac{H^{(1)}_m(ka)}{J_m(ka)}$$

$$\left[B^{\mathrm{I}}_m, B^{\mathrm{I}}_m \right] = ka \, H^{(1)\prime}_m(ka) \, H^{(1)}_m(ka) \frac{1}{(\cosh kd)^2} \int_{-d}^{0} (\cosh k(z+d))^2 \, dz$$

$$+ \left\{ -k \int_0^a r \, (J_m(kr))^2 dr \frac{\sinh kd}{\cosh kd} \frac{\cosh kh}{\cosh kd} \right.$$

$$-ka\ J'_m(ka)\ J_m(ka)\frac{1}{(\cosh\ kd)^2}\int_{-d}^{-(d-h)}(\cosh\ k(z+d))^2dz\Big\}\cdot\frac{H_m^{(1)}(ka)}{J_m(ka)}{}^2$$

$$\Big[B_m^{\mathrm{I}},\ B_m^{\mathrm{II}}\Big]_i=\Big\{-\int_o^a r\ J_m(kr)\ J_m(k_{mi}r)\ dr\ \Big(k_{mi}\frac{\cosh\ kh}{\cosh\ kd}\ \frac{\sinh\ k_{mi}h}{\cosh\ k_{mi}d}+k\frac{\sinh\ kh}{\cosh\ kd}\ \frac{\cosh\ k_{mi}h}{\cosh\ k_{mi}d}\Big)$$

$$-(k_{mi}aJ_m(ka)\ J'_m(k_{mi}a)+ka\ J'_m(ka)\ J_m(k_{mi}a))\frac{1}{\cosh\ kd\ \cosh\ k_{mi}d.}$$

$$\cdot\int_{-d}^{-(d-h)}\cosh\ k(z+d)\ \cosh\ k_{mi}(z+d)\ dz\Big\}\frac{H_m^{(1)}(ka)}{J_m(ka)}$$

$$\Big[A_m^{\mathrm{I}},\ B_m^{\mathrm{II}}\Big]_{ij}=-\frac{1}{I_m(\alpha_i a)}\int_o^a r\ J_m(k_{mj}r)\ I_m(\alpha_i r)\ dr\cdot\Big(k_{mj}\frac{\sinh\ k_{mj}h}{\cosh\ k_{mj}d}\cos\ \alpha_i h-\alpha_i\ \sin\ \alpha_i h\ \frac{\cosh\ k_{mj}h}{\cosh\ k_{mj}d}\Big)$$

$$-(k_{mj}a\ J'_m(k_{mj}a)+\alpha_i\ a\frac{I'_m(\alpha_i a)}{I_m(\alpha_i a)}J_m(k_{mj}a))\cdot\frac{1}{\cosh\ k_{mj}d}\int_{-d}^{-(d-h)}\cosh\ k_{mj}(z+d)\ \cos\ \alpha_i(z+d)\ dz$$

$$\Big[B_m^{\mathrm{II}},\ B_m^{\mathrm{II}}\Big]_{ij}=-k_{mi}\int_o^a r\ J_m(k_{mi}r)\ J_m(k_{mj}r)\ dr\ \frac{\sinh\ k_{mi}h}{\cosh\ k_{mi}d}\ \frac{\cosh\ k_{mj}h}{\cosh\ k_{mj}d}$$

$$-k_{mi}a\ J'_m(k_{mi}a)\ J_m(k_{mj}a)\ \frac{1}{\cosh\ k_{mi}d\ \cosh\ k_{mj}d}\cdot\int_{-d}^{-(d-h)}\cosh\ k_{mi}(z+d)\ \cosh\ k_{mj}\ (z+d)\ dz$$

$$\Big[A_m^{\mathrm{I}}\Big]_i=a\int_{-(d-h)}^{o}f_m^{\mathrm{I}}(z)\ \cos\ \alpha_i(z+d)\ dz+\frac{1}{I_m(\alpha_i a)}\int_o^a r\ f_m^{\mathrm{II}}(r)\ I_m(\alpha_i\ r)\ dr\ \cos\ \alpha_i h$$

$$\Big[B_m^{\mathrm{I}}\Big]=a\ H_m^{(1)}(ka)\frac{1}{\cosh\ kd}\int_{-(d-h)}^{o}f_m^{\mathrm{I}}(z)\ \cosh\ k(z+d)\ dz+\int_o^a r\ f_m^{\mathrm{II}}(r)\ J_m(kr)\ dr\frac{\cosh\ kh}{\cosh\ kd}\ \frac{H_m^{(1)}(ka)}{J_m(ka)}$$

$$\Big[B_m^{\mathrm{II}}\Big]_i=\int_o^a r\ f_m^{\mathrm{II}}(r)\ J_m(k_{mi}r)\ dr\ \frac{\cosh\ k_{mi}h}{\cosh\ k_{mi}d}$$

Formulae for the integrals $\int\cos\ \alpha_i(z+d)\ \cos\ \alpha_j(z+d)\ dz$, $\int r\ I_m(\alpha_i r)\ I_m(\alpha_j r)\ dr$, etc. are given in appendix B.

Appendix B.

$$\int\cos\ \alpha_i z\ \cos\ \alpha_j z\ dz=\begin{cases}\dfrac{1}{\alpha_i^2-\alpha_j^2}(\alpha_i\ \sin\ \alpha_i z\ \cos\ \alpha_j z-\alpha_j\ \cos\ \alpha_i z\ \sin\ \alpha_j z) & \text{for } \alpha_i\neq\alpha_j\\[2mm] \dfrac{z}{2}+\dfrac{1}{4\alpha_i}\sin\ 2\alpha_i z & \text{for } \alpha_i=\alpha_j\end{cases}$$

$$\int\cosh\ kz\ \cos\ \alpha_i z\ dz=\frac{1}{\alpha_i^2+k^2}(k\ \sinh\ kz\ \cos\ \alpha_i z+\alpha_i\ \cosh\ kz\ \sin\ \alpha_i z)$$

$$\int\cosh\ k_{mi}z\ \cosh\ k_{mj}z\ dz=\begin{cases}\dfrac{1}{k_{mi}^2-k_{mj}^2}(k_{mi}\ \cosh\ k_{mj}z\ \sinh\ k_{mi}z-k_{mj}\ \cosh\ k_{mi}z\ \sinh\ k_{mj}z) & \text{for } k_{mi}\neq k_{mj}\\[2mm] \dfrac{z}{2}+\dfrac{1}{2k_{mi}}\cosh\ k_{mi}z\ \sinh\ k_{mi}z & \text{for } k_{mi}=k_{mj}\end{cases}$$

$$\int r\ I_m(\alpha_i r)\ I_m(\alpha_j r)\ dr=\begin{cases}\dfrac{r}{\alpha_i^2-\alpha_j^2}\{-\alpha_j I_m(\alpha_i r)\ I_{m-1}\ (\alpha_j r)+\alpha_i I_{m-1}\ (\alpha_i r)\ I_m(\alpha_j r)\} & \text{for } \alpha_i\neq\alpha_j\\[2mm] \dfrac{r^2}{2}\{(I_m(\alpha_i r))^2-I_{m-1}(\alpha_i r)\ I_{m+1}(\alpha_i r)\} & \text{for } \alpha_i=\alpha_j\end{cases}$$

$$\int r\ J_m(kr)\ I_m(\alpha_i r)\ dr=\frac{r}{k^2+\alpha_i^2}\ \{\alpha_i J_m(kr)\ I_{m-1}(\alpha_i r)-k\ J_{m-1}\ (kr)\ I_m(\alpha_i r)\}$$

$$\int r\ J_m(k_{mi}r)\ J_m(k_{mj}r)\ dr=\begin{cases}\dfrac{r}{k_{mi}^2-k_{mj}^2}k_{mj}J_m(k_{mi}r)\ J_{m-1}\ (k_{mj}r)-k_{mi}J_{m-1}\ (k_{mi}r)\ J_m(k_{mj}r) & \text{for } k_{mi}\neq k_{mj}\\[2mm] \dfrac{r^2}{2}\ (J_m(k_{mi}r))^2-J_{m-1}(k_{mi}r)\ J_{m+1}(k_{mi}r)\} & \text{for } k_{mi}=k_{mj}\end{cases}$$

$$\int r^{m+1}\ I_m(\alpha_i r)\ dr=\frac{r^{m+1}}{\alpha_i}I_{m+1}(\alpha_i r)\qquad\qquad \int r^{m+1}\ J_m(kr)\ dr=\frac{r^{m+1}}{k}\ J_{m+1}(kr)$$